BENT FUGLEDE

CAPACITY AS A SUBLINEAR FUNCTIONAL GENERALIZING AN INTEGRAL

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CONTENTS

Synopsis

With a view at certain applications in potential theory a general study is made of a capacity c in the sense of a finite valued, increasing, and sublinear functional $\geqq 0$ defined on the cone of all finite valued, continuous functions $\geqq 0$ of compact support on a locally compact space X.

It is shown that any such capacity c is representable as the supremum of the family of all linear capacities (= positive Radon measures) majorized by c. Like in integration theory, c may be extended to a lower capacity c_* and an upper capacity c^*, both defined for arbitrary functions on X to $[0, +\infty]$. The main object is the investigation of certain function classes, closed with respect to c^*.

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BIANCO LUNOS BOGTRYKKERI A/S

Introduction

The notion of capacity is usually that of a set function (having some of the properties of a measure). For the investigation of the capacities in potential theory I have found it advantageous to view a capacity C primarily as a functional (with some of the properties of an integral) rather than a set function. Thus one avoids the separate consideration of several capacities with respect to different "weight functions" (usually hyperharmonic functions or potentials), cf. § 6.7.

In the present paper we treat a rather general notion of capacity as a sublinear functional. More precisely we begin (Chapter I) by considering a capacity as an increasing and *countably sublinear* mapping $C: \mathscr{F}^+(X) \to [0, +\infty]$. Here $\mathscr{F}^+(X)$ denotes the class of all functions on a set X into $[0, +\infty]$. When X is a topological space we call C an *upper capacity* if, in addition, the value $C(f)$ for any $f \in \mathscr{F}^+(X)$ is the infimum of $C(g)$ for $g \geqq f$ and g belonging to the class $\mathscr{G} = \mathscr{G}(X)$ of all lower semicontinuous functions $g \in \mathscr{F}^+(X)$.

Next we study (Chapter II), likewise under the name of capacity, an increasing *sublinear* functional $c: \mathscr{C}_0^+(X) \to [0, +\infty[$, where $\mathscr{C}_0^+(X)$ denotes the class of all continuous functions of compact support on a *locally compact* space X, and with finite values $\geqq 0$. As in the special (linear) case of a Radon measure (cf. CARTAN [8], BOURBAKI [2]) such a functional (capacity) c has natural extensions to a *lower capacity* c_* and an *upper capacity* c^*. The latter is also an upper capacity in the sense of Chapter I.

The main interest will be focussed upon the investigation of certain subclasses of $\mathscr{F}^+(X)$. In addition to the above class $\mathscr{G}$ we have the class $\mathscr{H}_0$ of all upper semicontinuous functions of finite values $\geqq 0$ and of compact support. By a process of closure defined in terms of the (upper) capacity in question we arrive at the basic function classes $\mathscr{G}^*$ and $\mathscr{H}_0^*$. In the special case of a Radon measure on a locally compact space which is countable at infinity, $\mathscr{G}^*$ and $\mathscr{H}_0^*$ consist of those functions (in $\mathscr{F}^+(X)$) which are μ-measurable and μ-integrable, respectively. In the general case the two classes are unrelated, $\mathscr{G}^*$ being the class of all quasi lower semicontinuous functions in $\mathscr{F}^+(X)$ whereas the particularly important class $\mathscr{H}_0^*$ consists of certain quasi upper semicontinuous functions in $\mathscr{F}^+(X)$, cf. § 2.5.

In § 5 it is shown by a standard application of the Hahn-Banach theorem that any capacity $c: \mathscr{C}_0^+(X) \to [0, +\infty[$ (in the sense of Chapter II) is representable as the "upper envelope" (supremum) of a certain family $\mathscr{S}$ of positive Radon measures on the given locally compact space X—a result announced earlier by Choquet [10]. This allows us (Theorem 6.2) to characterize $\mathscr{G}^* \cap \mathscr{H}_0^*$ as the class of those functions $f \in \mathscr{G}^* \cup \mathscr{H}_0^*$ for which $\int f d\mu$ is finite and continuous when considered as a function of $\mu \in \mathscr{S}$, using the vague topology. In the theory of the "energy capacity" with respect to a symmetric lower semicontinuous kernel $G: X \times X \to [0, +\infty]$—a principal domain of application for the present study—this result implies that G is *consistent* (cf. [14], [15] in the case of a positive definite kernel) if and only if every potential $G\mu$ of finite energy $\int G\mu d\mu$ is of class $\mathscr{H}_0^*$ (with respect to the upper energy capacity). Brief indications of this and other potential theoretic applications of the present study are given in §§ 3.8, 5.7, and 6.7. A systematic account of such applications is in preparation.

Notations

For any set X we denote by $\mathscr{P}(X)$ the set of all subsets of X, and by $\mathscr{F}(X)$, resp. $\mathscr{F}^+(X)$, the set of all functions on X with values in $[-\infty, +\infty]$, resp. $[0, +\infty]$. The indicator function for a set A is denoted by 1_A. The symmetric difference between two sets A, B is denoted by $A \triangle B$. Furthermore, $\boldsymbol{R}$ and $\boldsymbol{N}$ denote the real and the natural numbers, respectively.

The usual lattice operations (pointwise supremum and infimum) on $\mathscr{F}(X)$ or $\mathscr{F}^+(X)$ are often designated by the symbols $\vee$ and $\wedge$, respectively, and we write $f^+ = f \vee 0$, $f^- = (-f) \vee 0$. Moreover, we write $f_n \nearrow f$ (resp. $f_n \searrow f$) to signify that the sequence $(f_n)_{n \in \boldsymbol{N}}$ of functions $f_n \in \mathscr{F}(X)$ is pointwise increasing (resp. decreasing) with the pointwise supremum (resp. infimum) f.

All the usual indeterminate expressions involving extended real numbers are interpreted as 0. For example,

$$0 \cdot (\pm \infty) = 0, \ (+\infty) + (-\infty) = 0.$$

Thus the elementary algebraic operations with extended real numbers are always well defined. Note that the triangle inequality

$$|a - b| \leqq |a - c| + |c - b|$$

holds for arbitrary $a, b, c, \in [-\infty, +\infty]$.

Chapter I

Capacity as a Countably Sublinear Functional on $\mathscr{F}^+(X)$

1. Capacity on an Abstract Space

In the present section X denotes a fixed set (without topology).

1.1. **Definitions.** *By a capacity on X we understand, in this chapter, an increasing, countably sublinear (= positive homogeneous and countably subadditive) mapping C of $\mathscr{F}^+(X)$ into $[0, +\infty]$.*

Thus we should have, for $f, f_1, f_2, \ldots \in \mathscr{F}^+(X)$,

(C₁) $[f_1 \leqq f_2] \Rightarrow [C(f_1) \leqq C(f_2)]$,

(C₂) $C(af) = a\, C(f)$ for $a \in [0, +\infty[$,

(C₃) $C(\sum_{n \in \boldsymbol{N}} f_n) \leqq \sum_{n \in \boldsymbol{N}} C(f_n)$.

Note that $C(0) = 0$ on account of (C₂).

Given a capacity C on X we put, for any extended real valued function $f \in \mathscr{F}(X)$,

$$\tilde{C}(f) := C(|f|).$$

Taking into account our conventions regarding indeterminate expressions (see Notations above) we obtain for $f, f_1, f_2 \in \mathscr{F}(X)$ and $a \in]-\infty, +\infty[$

$$\tilde{C}(af) = |a|\, \tilde{C}(f),$$

$$\tilde{C}(f_1 + f_2) \leqq \tilde{C}(f_1) + \tilde{C}(f_2).$$

It follows that $\tilde{C}(f_1 - f_2) = C(|f_1 - f_2|)$ defines a pseudometric on $\mathscr{F}(X)$ (in particular on $\mathscr{F}^+(X)$), to which we shall refer as the *C-metric* on $\mathscr{F}(X)$ (resp. $\mathscr{F}^+(X)$). Two functions $f_1, f_2 \in \mathscr{F}(X)$ are called *C-equivalent* (or just equivalent) if $C(|f_1 - f_2|) = 0$.

1.2. *The associated set function.* From the functional C on $\mathscr{F}^+(X)$ we derive a set function, likewise denoted by C, defined for all subsets of X by

$$C(A) = C(1_A).$$

Thus our capacity $C: \mathscr{F}^+(X) \to [0, +\infty]$ induces an increasing and countably subadditive set function $C: \mathscr{P}(X) \to [0, +\infty]$ for which $C(\emptyset) = 0$. Explicitly we have, for $A_1, A_2, \ldots \subset X$,

$$[A_1 \subset A_2] \Rightarrow [C(A_1) \leqq C(A_2)],$$

$$C(\bigcup_{n \in \boldsymbol{N}} A_n) \leqq \sum_{n \in \boldsymbol{N}} C(A_n),$$

$$C(\emptyset) = 0.$$

A set function with these properties was studied in [17], likewise under the name of a capacity on X. Concepts and results from [17] will be carried over freely to the present case of the set function associated with a capacity in the sense of § 1.1 above.

In particular, a property $P[x]$ is said to hold *quasi everywhere* (q. e.) in a set $A \subset X$ if $C(\{x \in A \mid \text{non } P[x]\}) = 0$. If $A = X$ we may write simply q. e. (in place of q. e. in X).

1.3. **Lemma:** *For functions $f, f_1, f_2 \in \mathscr{F}^+(X)$ we have:*

(a) $[f(x) = 0 \text{ q.e.}] \Leftrightarrow [C(f) = 0]$.

(b) $[f_1(x) \leqq f_2(x) \text{ q.e.}] \Leftrightarrow [C((f_1 - f_2)^+) = 0] \Rightarrow [C(f_1) \leqq C(f_2)]$.

(c) $[C(f) < +\infty] \Rightarrow [f(x) < +\infty \text{ q.e.}]$.

Proof. (a) Let $E: = \{x \in X \mid f(x) > 0\}$. Then

$$f \leqq 1_E + 1_E + \ldots; \qquad 1_E \leqq f + f + \ldots,$$

from which the assertion follows by use of (C_3). As to (b), write $f: = (f_1 - f_2)^+$, and apply (a). Next observe that $f_1 \leqq f_2 + f$, and hence by (C_1), (C_3): $C(f_1) \leqq C(f_2 + f) \leqq C(f_2) + C(f) = C(f_2)$ if $C(f) = C((f_1 - f_2)^+) = 0$. To establish (c) let $E_t: = \{x \in X \mid f(x) \geqq t\}$ for any $t \in]0, +\infty]$. Then $t 1_{E_t} \leqq f$ for every finite t, and hence by (C_1), (C_2)

$$C(E_{+\infty}) \leqq C(E_t) = t^{-1} C(t 1_{E_t}) \leqq t^{-1} C(f),$$

from which the result follows for $t \to +\infty$. ▌

Corollary 1. *Let $f \in \mathscr{F}^+(X)$, $t \in]0, +\infty[$, $A \subset X$, and suppose that*

$$f(x) \geqq t \qquad \text{q.e. in } A.$$

Then $C(A) \leqq t^{-1} C(f)$.

(In fact, $1_A \leqq t^{-1} f$ quasi everywhere.)

Corollary 2. *Two functions $f_1, f_2 \in \mathscr{F}(X)$ are C-equivalent if and only if $f_1(x) = f_2(x)$ q.e.*

(Apply Lemma 1.3. (a) to $f: = |f_1 - f_2|$.)

1.4. *Quasi uniform convergence.* A sequence of functions $f_n \in \mathscr{F}(X)$ is said to converge *quasi uniformly* to a function $f \in \mathscr{F}(X)$ if there exists for every $\varepsilon > 0$ a set $\omega \subset X$ with $C(\omega) < \varepsilon$ such that f_n converges uniformly to f on $\complement\omega$ as $n \to \infty$. In the affirmative case we clearly have $f_n(x) \to f(x)$ pointwise q. e.

Theorem. *If a sequence of functions $f_n \in \mathscr{F}(X)$ converges in the C-metric to a function $f \in \mathscr{F}(X)$, then there exists a subsequence of (f_n) which converges quasi uniformly to f.*

Proof. Passing to a suitable subsequence we may suppose that $C(|f_n - f|) < 4^{-n}$ for $n = 1, 2, \ldots$. Writing

$$M_n := \{x \in X \,|\, |f_n(x) - f(x)| > 2^{-n}\},$$
$$N_p := \bigcup_{n > p} M_n,$$

we get, from Cor. 1 to Lemma 1.3, $C(M_n) \leqq 2^n C(|f_n - f|) < 2^{-n}$, and hence $C(N_p) \leqq \sum_{n>p} 2^{-n} = 2^{-p}$. Clearly $f_n(x) \to f(x)$ uniformly on $\complement N_p$ for each p, hence quasi-uniformly. ∎

1.5. *The Banach space L(C).* Like in integration theory one might consider the subset $\mathscr{L} \subset \mathscr{F}(X)$ consisting of all $f \in \mathscr{F}(X)$ with $C(|f|) < +\infty$ (and hence $|f(x)| < +\infty$ q.e.). The quotient space $L = L(C)$ of $\mathscr{L}$ with respect to C-equivalence is a vector space (unlike $\mathscr{L}$ itself), and the mapping $f \mapsto C(|f|)$ of $\mathscr{L}$ into $[0, +\infty[$ induces a norm on $L(C)$. By the standard Riesz-Fischer technique it can be easily shown that $L(C)$ is complete in this norm, i.e. $L(C)$ is a *Banach space.* (This result, however, will not be used in the sequel.)

1.6. *Souslin functions. Capacitability.* Let $\mathscr{H}$ denote a subset of $\mathscr{F}^+(X)$ containing 0 and stable under countable infimum. By an $\mathscr{H}$-Souslin function $f \in \mathscr{F}^+(X)$ we understand a function which can be obtained from functions of class $\mathscr{H}$ by application of Souslin's operation (A) as described e.g. in Choquet [11] (with the obvious changes caused by our consideration of the function lattice $\mathscr{F}^+(X)$ instead of the lattice $\mathscr{P}(X)$ of all subsets of X).

The class of all $\mathscr{H}$-Souslin functions is stable under countable supremum or infimum and contains $\mathscr{H}$.

Consider now a capacity C on X, or equally well any increasing mapping

$$C\colon \mathscr{F}^+(X) \to [0, +\infty].$$

According to Choquet [12], a function $f \in \mathscr{F}^+(X)$ is called *$(C,\mathscr{H})$-capacitable* if

$$C(f) = \sup\{C(h) \,|\, h \in \mathscr{H},\, h \leqq f\}.$$

We quote the main theorem of capacitability in the present abstract case:

Theorem (CHOQUET [12]). *Suppose that the increasing functional* C: $\mathscr{F}^+(X) \to [0, +\infty]$ *is sequentially order continuous from above on* $\mathscr{H}$, *and sequentially order continuous from below on all of* $\mathscr{F}^+(X)$:

$$[h_n \searrow h, h_n \in \mathscr{H}] \Rightarrow [C(h_n) \to C(h)],$$
$$[f_n \nearrow f, f_n \in \mathscr{F}^+(X)] \Rightarrow [C(f_n) \to C(f)].$$

Then every $\mathscr{H}$*-Souslin function* $f \in \mathscr{F}^+(X)$ *is* $(C,\mathscr{H})$*-capacitable.*

2. Some Basic Classes of Functions

In the sequel X denotes a Hausdorff[1] topological space. Writing l.s.c. and u.s.c. for lower and upper semicontinuous, respectively, we shall consider the following subclasses of the class $\mathscr{F}^+(X)$ of all functions $f: X \to [0, +\infty]$:

$$\begin{aligned}
&\mathscr{G}\ (= \mathscr{G}(X)) = \{f \in \mathscr{F}^+(X) \mid f \text{ is l.s.c.}\},\\
&\mathscr{H} = \{f \in \mathscr{F}^+(X) \mid f \text{ is u.s.c. and finite}\},\\
&\mathscr{H}_0 = \{f \in \mathscr{H} \mid f \text{ has compact support}\},\\
&\mathscr{C}^+ = \mathscr{G} \cap \mathscr{H} = \{f \in \mathscr{F}^+(X) \mid f \text{ is continuous and finite}\},\\
&\mathscr{C}_0^+ = \mathscr{G} \cap \mathscr{H}_0 = \{f \in \mathscr{C}^+ \mid f \text{ has compact support}\}.
\end{aligned}$$

2.1. *The closed classes.* $\mathscr{G}^*, \mathscr{H}^*, \mathscr{H}_0^*$. Let C denote a given capacity on X in the sense of § 1.1. We denote by $\mathscr{G}^*$, $\mathscr{H}^*$, and $\mathscr{H}_0^*$ the *closures* of $\mathscr{G}$, $\mathscr{H}$, and $\mathscr{H}_0$, respectively, in the C-metric topology on $\mathscr{F}^+(X)$. Thus we have, for $f \in \mathscr{F}^+(X)$,

$$[f \in \mathscr{G}^*] \Leftrightarrow [\inf\{C(|f - \varphi|) \mid \varphi \in \mathscr{G}\} = 0],$$

and similarly with $\mathscr{H}$ or $\mathscr{H}_0$ in place of $\mathscr{G}$ (and $\mathscr{H}^*$ or $\mathscr{H}_0^*$ in place of $\mathscr{G}^*$). Clearly $\mathscr{H}_0^* \subset \mathscr{H}^*$. Every function of class $\mathscr{H}^*$ is finite q.e. Each of the 3 classes is a convex cone which is saturated with respect to C-equivalence (§ 1.1.) within $\mathscr{F}^+(X)$.

2.2. **Theorem.** $\mathscr{G}^*$ *is stable under countable supremum and finite infimum.* $\mathscr{H}^*$ *and* $\mathscr{H}_0^*$ *are stable under countable infimum and finite supremum.*

Proof. Follows easily from the corresponding properties of $\mathscr{G}$, $\mathscr{H}$, $\mathscr{H}_0$ in view of the inequalities

$$\left.\begin{array}{l} |\sup\limits_n f_n - \sup\limits_n \varphi_n| \\ |\inf\limits_n f_n - \inf\limits_n \varphi_n| \end{array}\right\} \leqq \sup_n |f_n - \varphi_n| \leqq \sum_n |f_n - \varphi_n|$$

for finite or infinite families (f_n) and (φ_n) of functions of class $\mathscr{F}^+(X)$. ∎

[1]) The Hausdorff separation property is needed only in contexts involving *compact* subsets of X, e.g. in connection with functions of compact support, thus in particular for the classes $\mathscr{H}_0, \mathscr{C}_0^+$, and $\mathscr{H}_0^*$.

2.3. **Lemma.** *If $g \in \mathscr{G}^*$, $h \in \mathscr{H}^*$, then*

$$(g-h)^+ \in \mathscr{G}^*, \qquad (h-g)^+ \in \mathscr{H}^*.$$

Similarly with $\mathscr{H}_0^*$ in place of $\mathscr{H}^*$.

Proof. Follows easily from the corresponding properties of $\mathscr{G}$ and $\mathscr{H}$ (resp. $\mathscr{H}_0$) in view of the inequality

$$|(g-h)^+ - (\varphi - \psi)^+| \leqq |g - \varphi| + |h - \psi|$$

for functions g, h, φ, ψ of class $\mathscr{F}^+(X)$. ▌

2.4. **Lemma,** *Let $A \subset X$. Then A is quasi open (quasi closed) if and only if $1_A \in \mathscr{G}^*$ ($1_A \in \mathscr{H}^*$). Moreover, A is quasi compact if and only if $1_A \in \mathscr{H}_0^*$.*

Proof. As to the notions quasi open, etc., see [17, § 2], to be applied here to the set function $C(A) = C(1_A)$ associated with the given capacity functional $C : \mathscr{F}^+(X) \to [0, +\infty]$ as described in § 1.2. The "only if" part of the lemma is obvious since

$$\begin{aligned} &[A \text{ open (closed)}] \Leftrightarrow [1_A \in \mathscr{G}(1_A \in \mathscr{H})], \\ &[A \text{ compact}] \qquad \Leftrightarrow [1_A \in \mathscr{H}_0]. \end{aligned}$$

Conversely, let $1_A \in \mathscr{H}^*$ (resp. $\mathscr{H}_0^*$), and choose $\varphi \in \mathscr{H}$ (resp. $\mathscr{H}_0$) so that $C(|1_A - \varphi|) < \varepsilon$. The set $E\colon = \{x \in X \,|\, \varphi(x) \geqq \frac{1}{2}\}$ is closed (resp. compact), and $|1_A - \varphi| \geqq \frac{1}{2}$ on $A \,\Delta E$, that is,

$$1_{A \Delta E} \leqq 2|1_A - \varphi|.$$

Consequently, $C(A \,\Delta E) \leqq 2C(|1_A - \varphi|) < 2\varepsilon$. The proof is quite similar in the case $1_A \in \mathscr{G}^*$. ▌

2.5. **Theorem.** *Let $f \in \mathscr{F}^+(X)$. Then*

(a) *$f \in \mathscr{G}^*$ if and only if f is quasi l.s.c.*
(b) *The following propositions are equivalent:*
   (i) *$f \in \mathscr{H}_0^*$.*
   (ii) *f is quasi u.s.c. and has a majorant of class $\mathscr{H}_0^*$.*
   (iii) *f is quasi u.s.c., and* $\inf\{C((f-h)^+) \,|\, h \in \mathscr{H}_0\} = 0$.

Proof. As to the notions of quasi continuity and quasi semicontinuity, see [17, § 3]. Any function f of class $\mathscr{G}^*$ (resp. $\mathscr{H}^*$) is by definition a limit in the C-metric topology (§ 1.1) of a sequence of functions of class $\mathscr{G}$ (resp. $\mathscr{H}$), and hence f is quasi l.s.c. (resp. quasi u.s.c.) according to Theorem 1.4, because the quasi uniform limit of a sequence of quasi l.s.c. (resp. quasi u.s.c.) functions is a function of the same kind ([17, th. 3.2]). This establishes the "only if" part of (a) and the implication (i) $\Rightarrow$ (ii) (even with $\mathscr{H}^*$ in place of $\mathscr{H}_0^*$).

Next suppose that f is quasi l.s.c. For every $n \in \boldsymbol{N}$ the function $f_n := f \wedge n$ is quasi l.s.c. and bounded. We propose to show that $f_n \in \mathscr{G}^*$ for all $n \in \boldsymbol{N}$, and consequently that $f = \sup_n f_n \in \mathscr{G}^*$ according to Theorem 2.2. By definition there are sets ω of arbitrary small $C(\omega)$ such that f_n is l.s.c. relatively to $\complement\omega$. Since $0 \leqq f_n \leqq n$ there is a l.s.c. function φ_n such that $0 \leqq \varphi_n \leqq n$ which agrees with f_n on $\complement\omega$.[2] Clearly $C(|f_n - \varphi_n|) \leqq nC(\omega)$ is as small as we please.

As to (b), the implication (ii) $\Rightarrow$ (iii) is obvious (even without quasi u.s.c.), since for any $g \in \mathscr{H}_0^*$ such that $g \geqq f$ and for any $h \in \mathscr{H}_0$

$$(f-h)^+ \leqq (g-h)^+ \leqq |g-h|.$$

Finally suppose that (iii) holds. Since $(f-h)^+ = f - f \wedge h$, this implies that f may be approximated in the C-metric topology by functions of the form $f \wedge h$ with $h \in \mathscr{H}_0$. Since $f \wedge h$ is quasi u.s.c. and $\leqq h \in \mathscr{H}_0$, it suffices to prove that any quasi u.s.c. function $f \in \mathscr{F}^+(X)$ having a majorant $h \in \mathscr{H}_0$, is of class $\mathscr{H}_0^*$. By definition there are sets ω of arbitrary small $C(\omega)$ such that the restriction of f to $\complement\omega$ is u.s.c. The u.s.c. envelope φ of $f \cdot 1_{\complement\omega}$ is $\leqq h$, hence $\varphi \in \mathscr{H}_0$, and agrees with f on $\complement\omega$. Since h, and hence f, is bounded, say $\leqq a$, we find that $C(|f-\varphi|) = C(1_\omega |f-\varphi|) \leqq aC(\omega)$ is as small as we please. ∎

Definition. *A function $f \in \mathscr{F}(X)$ is called semibounded if*

$$\inf_{t \geqq 0} C((|f|-t)^+) = 0.$$

Corollary. *A function $f \in \mathscr{F}^+(X)$ is of class $\mathscr{H}_0^*$ if and only if f is quasi u.s.c., semibounded, and has the further property*

$$\inf\{C(f \cdot 1_{\complement K}) | K \text{ compact}\} = 0.$$

This follows by use of (iii) of the above theorem since any $h \in \mathscr{H}_0$ has a majorant of the form $t \cdot 1_K$ with $t \in [0, +\infty[$, K compact; and since any such function $t \cdot 1_K$ is of class $\mathscr{H}_0$.

Note also that the above condition $\inf C(f \cdot 1_{\complement K}) = 0$ is (necessary and) sufficient for a function $f \in \mathscr{H}^*$ to belong to $\mathscr{H}_0^*$. This appears from the estimate

[2]) For instance take for φ_n the l.s.c. envelope of the function which equals f_n in $\complement\omega$, and n in ω.

$$|f-h\cdot 1_K| \leqq f\cdot 1_{\complement K} + |f-h|$$

with $h \in \mathscr{H}$, K compact, and hence $h\cdot 1_K \in \mathscr{H}_0$.

Remark. In the case of an upper capacity C (§ 3.1 below) on our Hausdorff space X, any function of class $\mathscr{H}_0^*$, and more generally any function $f \in \mathscr{F}(X)$ such that $\inf\{C(|f|\cdot 1_{\complement K})\,|\,K \text{ compact}\} = 0$, has in particular the *quasi limit* 0 at infinity, in the sense that there are open sets $\omega \subset X$ with $C(\omega)$ as small as we please such that the restriction of f to $\complement\omega$ vanishes at infinity in the closed, hence locally compact, subspace $\complement\omega$. This follows from [17, § 3.6.] since any function f as stated is the limit in the C-metric topology (hence also the quasi uniform limit) of a sequence of functions of compact support.

Similarly, any function $f \in \mathscr{H}_0^*$, and more generally any semibounded function $f \in \mathscr{F}(X)$, is in particular *quasi bounded* in the sense that there are sets $\omega \subset X$ with $C(\omega)$ as small as we please such that f is bounded on $\complement\omega$. Note also that any function $f \in \mathscr{F}(X)$ such that $C(|f|) < +\infty$, is quasi bounded, and that any quasi bounded function is finite quasi everywhere. (This follows from the proof of Lemma 1.3 (c).)

2.6. **Theorem.** *Let $f \in \mathscr{F}^+(X)$, and consider the following statements*:

(i) $f \in \mathscr{H}^*$.
(ii) *f is quasi u.s.c. and has a majorant of class $\mathscr{H}^*$.*
(iii) *f is quasi u.s.c., and* $\inf\{C((f-h)^+)\,|\,h \in \mathscr{H}\} = 0$.
(iv) *$\varphi f \in \mathscr{H}_0^*$ for every $\varphi \in \mathscr{H}_0$ (or just for every $\varphi \in \mathscr{C}_0^+$, or every $\varphi = 1_K$ with K compact).*

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv). *Conversely* (iv) *implies* (i) *if X is locally compact and countable at infinity.*

Proof. For the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) see the corresponding parts of the proof of Theorem 2.5. For any $\varphi \in \mathscr{H}_0$, say with $\varphi \leqq a$, the estimate

$$(\varphi f - \varphi h)^+ = \varphi\cdot(f-h)^+ \leqq a(f-h)^+$$

allows us to conclude from (iii) that $\varphi f \in \mathscr{H}_0^*$ (by use of (iii) of Theorem 2.5.) because $\varphi h \in \mathscr{H}_0$ (when $\varphi \in \mathscr{H}_0$ and $h \in \mathscr{H}$), and φf is quasi u.s.c. along with f (when φ is u.s.c.). Finally, the validity of (iv) for all $\varphi = 1_K$ with K compact implies the same for any $\varphi \in \mathscr{H}_0$ (thus in particular for any $\varphi \in \mathscr{C}_0^+$) because the relation $\varphi f = \varphi\cdot(1_K f)$ (with K = the compact support of φ) shows that φf is of class $\mathscr{H}_0^*$ as product of $\varphi \in \mathscr{H}_0$ and

$1_K f \in \mathscr{H}_0^*$. Thus it remains to prove that f is of class $\mathscr{H}^*$ whenever $\varphi f \in \mathscr{H}_0^*$ (or equivalently $\varphi f \in \mathscr{H}^*$) for all $\varphi \in \mathscr{C}_0^+$, assuming that X is locally compact and countable at infinity. It is well known that there exists on such a space a partition of unity $(\varphi_n)_{n \in \boldsymbol{N}}$ of class $\mathscr{C}_0^+$ with the property that every compact subset of X meets the support of at most finitely many φ_n. For given $\varepsilon > 0$ choose functions $h_n \in \mathscr{H}$ so that $C(|\varphi_n f - h_n|) < \varepsilon/2^n$. Clearly we may assume that h_n vanishes outside the support of φ_n. The function $h := \sum_{n \in \boldsymbol{N}} h_n$ is then likewise of class $\mathscr{H}$ because the sum is finite in any compact neighbourhood of a point of X. For $f - h = \sum_{n \in \boldsymbol{N}} (\varphi_n f - h_n)$ we find $C(|f-h|) < \varepsilon$. ▮

Corollary. *On a locally compact space X which is countable at infinity a function $f \in \mathscr{F}^+(X)$ is of class $\mathscr{H}^*$ if and only if f is quasi u.s.c. and locally semibounded in the sense that $f \cdot 1_K$ is semibounded for every compact set K (or equivalently for some neighbourhood K of every point of X).*

3. Upper Capacity

3.1. **Definition.** *By an upper capacity on a topological space X we understand a functional $C: \mathscr{F}^+(X) \to [0, +\infty]$ which, in addition to* (C_1), (C_2), (C_3) *of* § 1.1, *has the following property for every $f \in \mathscr{F}^+(X)$:*

$$(C_4) \qquad C(f) = \inf\{C(g) \mid g \in \mathscr{G},\ g \geqq f\}.$$

The associated *set function* $C: \mathscr{P}(X) \to [0, +\infty]$ (§ 1.2) is then an *outer capacity* in the sense of [17, § 1.5] because

$$C(A) = \inf\{C(G) \mid G \text{ open},\ G \supset A\}$$

for every set $A \subset X$. In fact, let $g \in \mathscr{G}$, $g \geqq 1_A$, and let $0 < t < 1$. Then

$$G := \{x \in X \mid g(x) > t\}$$

is open and contains A. According to Cor. 1 to Lemma 1.3 (or directly), $C(G) \leqq t^{-1} C(g)$, and here the right hand member may be taken as close as we please to $C(1_A) = C(A)$ by virtue of (C_4)

In the sequel we always suppose that $C: \mathscr{F}^+(X) \to [0, +\infty]$ denotes an *upper* capacity, and moreover that the topological space X is a Hausdorff space (at least in contexts involving compactness).

3.2. **Lemma.** *In the case of an upper capacity C we have, for any $f \in \mathscr{F}^+(X)$:*

$$[f \in \mathscr{G}^*] \Leftrightarrow [\inf\{C(\varphi - f) \mid \varphi \in \mathscr{G},\ \varphi \geqq f\} = 0],$$
$$[f \in \mathscr{H}^*] \Leftrightarrow [\inf\{C(f - \varphi) \mid \varphi \in \mathscr{H},\ \varphi \leqq f\} = 0],$$
$$[f \in \mathscr{H}_0^*] \Leftrightarrow [\inf\{C(f - \varphi) \mid \varphi \in \mathscr{H}_0,\ \varphi \leqq f\} = 0].$$

Proof. By definition (§ 2.1), the implication $\Leftarrow$ holds (for any capacity). Conversely, let $f \in \mathscr{G}^*$ and $\varepsilon > 0$ be given. Choose $\psi \in \mathscr{G}$ so that $C(|f-\psi|) < \varepsilon/2$, and next $g \in \mathscr{G}$ with $g \geqq |f-\psi|$ so that $C(g) < \varepsilon/2$. Then we may use $\varphi = \psi + g$ $(\in \mathscr{G})$ since $\varphi \geqq \psi + |f - \psi| \geqq f$, $\varphi - f = g + (\psi - f)$, and hence

$$C(\varphi - f) \leqq C(g) + C(|\psi - f|) < \varepsilon.$$

In the case $f \in \mathscr{H}^*$ (or $\mathscr{H}_0^*$), proceed similarly, now with $\psi \in \mathscr{H}$ (or $\mathscr{H}_0$), and take $\varphi = (\psi - g)^+$. ▌

3.3. **Theorem.** *In the case of an upper capacity C on a locally compact space X, $\mathscr{G}^* \cap \mathscr{H}_0^*$ is the closure of $\mathscr{G} \cap \mathscr{H}_0 = \mathscr{C}_0^+$ in the C-metric topology on $\mathscr{F}^+(X)$.*

Proof. We shall prove that any function $f \in \mathscr{G}^* \cap \mathscr{H}_0^*$ can be approximated in the C-metric (§ 1.1) by functions $\varphi \in \mathscr{C}_0^+$. For any $\varepsilon > 0$ there exist, by the preceding lemma, functions $g \in \mathscr{G}$ and $h \in \mathscr{H}_0$ such that $h \leqq f \leqq g$, $C(g-f) < \varepsilon/2$, $C(f-h) < \varepsilon/2$, and hence $C(g-h) < \varepsilon$. According to Lemma 3.4 below there exists $\varphi \in \mathscr{C}_0^+$ such that $h \leqq \varphi \leqq g$. It follows that $|f - \varphi| \leqq g - h$, and hence $C(|f-\varphi|) < \varepsilon$. ▌

3.4. **Lemma.** *On a locally compact space X, let $g \in \mathscr{G}$, $h \in \mathscr{H}_0$, and suppose that $h \leqq g$. Then there exists $\varphi \in \mathscr{C}_0^+$ such that $h \leqq \varphi \leqq g$.*

Proof. This follows from the "*between theorem*", due to H. Hahn in the metrizable case (for a simple proof see Hausdorff [18]), and to H. Tong [20] in the most general case, viz. that of a normal space. The between theorem asserts that, if h is u.s.c. and g is l.s.c. on a normal space, and if $h \leqq g$, then there exists a continuous function φ such that $h \leqq \varphi \leqq g$. In the special case of a compact space X a simple direct proof can easily be given (cf. Bourbaki [1, 1. ed., exerc. 27, p. 72]), and this leads to the above lemma by compactification as follows (in the non-compact case):

Let $\hat{X} = X \cup \{\infty\}$ be the 1 point compactification of the locally compact space X, and define extensions $\hat{g}$, $\hat{h}$ of g, h from X to $\hat{X}$ by putting $\hat{g}(\infty) = \hat{h}(\infty) = 0$. Then $\hat{g}$ is l.s.c., and $\hat{h}$ is u.s.c. According to the between theorem for the compact space $\hat{X}$ there exists a continuous function $\hat{\varphi}$ on $\hat{X}$ such that $\hat{h} \leqq \hat{\varphi} \leqq \hat{g}$ (hence $\hat{\varphi} \geqq 0$). Let φ denote the restriction of $\hat{\varphi}$ to X, and put $a = \sup_{x \in X} h(x)$ $(< +\infty)$. Replacing if necessary φ by $\varphi \wedge a$ we may suppose φ finite. Since $h \in \mathscr{H}_0$ we may achieve that φ has compact support, hence $\varphi \in \mathscr{C}_0^+$, replacing if necessary φ by $\varphi\psi$, where $\psi \in \mathscr{C}_0^+(X)$ is so chosen that $\psi = 1$ on the support of h, and $\psi \leqq 1$ everywhere. ▌

Remark. The assumption that the space X be locally compact is easily shown to be necessary in Lemma 3.4 as well as in Theorem 3.3.

3.5. **Definition.** *An upper capacity $C : \mathscr{F}^+(X) \to [0, +\infty]$ will be called locally finite if it has one of the following equivalent properties:*

(i) $C(\{x\}) < +\infty$ *for every point* $x \in X$.
(ii) *Every point of X has a neighbourhood of finite capacity.*
(iii) $C(K) < +\infty$ *for every compact (or quasi compact) set* $K \subset X$.
(iv) $C(h) < +\infty$ *for every* $h \in \mathscr{H}_0$ (or $\mathscr{H}_0^*$).

3.6. **Theorem.** *Suppose that X is locally compact, and that the upper capacity C is locally finite. Then*

(a) *For any downward directed family of functions* $h_\alpha \in \mathscr{H}_0$

$$C(\inf_\alpha h_\alpha) = \inf_\alpha C(h_\alpha).$$

(b) *The same holds with $\mathscr{H}_0$ replaced by the larger class of all u.s.c. functions of class $\mathscr{H}_0^*$.*
(c) *For any decreasing sequence of functions* $h_n \in \mathscr{H}_0^*$

$$C(\inf_n h_n) = \inf_n C(h_n).$$

Proof. (a) Let $h = \inf_\alpha h_\alpha$. For any $t > C(h)$ there exists by (C_4), $g \in \mathscr{G}$ such that $g \geqq h$, $C(g) < t$. For such a function g, the downward directed family of functions $(h_\alpha - g)^+ \in \mathscr{H}_0$ converges pointwise to 0. According to Dini's theorem the convergence is uniform. For every $\varepsilon > 0$ there is, therefore, an index α such that $(h_\alpha - g)^+ < \varepsilon$ everywhere. We may take $\alpha \succcurlyeq \beta$ where β denotes a fixed index. Denoting by K the compact support of h_β, we obtain $h_\alpha \leqq g + \varepsilon \cdot 1_K$, and hence

$$C(h_\alpha) \leqq C(g + \varepsilon \cdot 1_K) \leqq C(g) + \varepsilon C(K) < t$$

by suitable choice of ε and next of $\alpha \succcurlyeq \beta$. This implies $\inf C(h_\alpha) < t$, and consequently $\inf C(h_\alpha) \leqq C(h)$. The converse inequality is obvious.

(b) In the slightly more general case where each h_α is u.s.c. and of class $\mathscr{H}_0^*$ we choose for some fixed index β and for any $\varepsilon > 0$ a function $\varphi \in \mathscr{H}_0$, $\varphi \leqq h_\beta$, so that $C(h_\beta - \varphi) < \varepsilon$. Since $h_\alpha \wedge \varphi \in \mathscr{H}_0$ decreases to $h \wedge \varphi$ (where again $h := \inf_\alpha h_\alpha$), we obtain from (a)

$$C(h \wedge \varphi) = \inf_\alpha C(h_\alpha \wedge \varphi),$$

and hence there is an index $\alpha \geqslant \beta$ such that

$$C(h_\alpha \wedge \varphi) < C(h \wedge \varphi) + \varepsilon \leqq C(h) + \varepsilon,$$
$$C(h_\alpha) \leqq C(h_\alpha \wedge \varphi) + C((h_\alpha - \varphi)^+) < C(h) + 2\varepsilon.$$

(c) Given $\varepsilon > 0$ choose $\varphi_n \in \mathscr{H}_0$ so that $\varphi_n \leqq h_n$ and $C(h_n - \varphi_n) < \varepsilon/2^n$ (Lemma 3.2), and put

$$f_n := \varphi_1 \wedge \ldots \wedge \varphi_n, \quad f := \inf_n \varphi_n = \inf_n f_n.$$

Then $(f_n)_{n \in \boldsymbol{N}}$ is a decreasing sequence of functions of class $\mathscr{H}_0$, and $f \leqq \inf_n h_n$. According to (a) above,

$$C(f_n) \leqq C(f) + \varepsilon \leqq C(\inf_n h_n) + \varepsilon$$

for all sufficiently large n. Since $(h_n)_{n \in \boldsymbol{N}}$ is decreasing, we have for every $n \in \boldsymbol{N}$

$$0 \leqq h_n - f_n \leqq \sup_{p \leqq n} (h_p - \varphi_p) \leqq \sum_{p \leqq n} (h_p - \varphi_p).$$

Consequently,

$$C(h_n) \leqq C(f_n) + C(h_n - f_n) \leqq C(f_n) + \varepsilon \leqq C(\inf_n h_n) + 2\varepsilon$$

for n sufficiently large. ▎

3.7. *Use of the quasi Lindelöf principle* (cf. Doob [13], Fuglede [17]). Suppose now that the Hausdorff space X has a *countable base* of open sets, and that the capacity C is *sequentially order continuous from below* (on $\mathscr{F}^+(X)$, cf. Theorem 1.6). Further let there be given on X a *new topology* τ, *compatible with* C in the sense of [17, § 4.3] (more precisely: compatible with the quase topology determined by the set function $A \mapsto C(A) := C(1_A)$ associated with A).

Under these hypotheses we have established in [17, § 4.4] the validity of the "*quasi Lindelöf principle*" of Doob [13]. Formulated for functions it states that any family (f_α) of τ-l.s.c. (resp. τ-u.s.c.) functions $f_\alpha \in \mathscr{F}(X)$ contains a countable subfamily $(f_{\alpha_n})_{n \in \boldsymbol{N}}$ whose pointwise supremum (resp. infimum) equals that of the given family quasi everywhere.—If (f_α) is upward (resp. downward) directed, we may of course achieve that the sequence (f_{α_n}) becomes increasing (resp. decreasing).

Theorem. *Under the hypotheses specified above we have* (a) $C(\sup_\alpha f_\alpha) = \sup_\alpha C(f_\alpha)$ *for any upward directed family of τ-l.s.c. functions $f_\alpha \in \mathscr{F}^+(X)$.*

(b) *If moreover X is locally compact and C a locally finite, upper capacity, then* $C(\inf_{\alpha} f_{\alpha}) = \inf_{\alpha} C(f_{\alpha})$ *for any downward directed family of τ-u.s.c. functions f_{α} having majorants of class $\mathscr{H}_0^*$ (and hence being themselves of class $\mathscr{H}_0^*$).*

Proof. The functions f_{α} are quasi l.s.c., respectively quasi u.s.c. (and hence of class $\mathscr{H}_0^*$ by Theorem 2.5), by virtue of the required compatibility between the new topology τ and the quasi topology determined by C, see [17, Lemma 4.3]. In view of the quasi Lindelöf principle stated above, statement (a) now follows from the assumed sequential order continuity of C from below, and (b) from Theorem 3.6 (c). ∎

3.8. *Example.* The first results of the type of Theorem 3.7 were established by Brelot [5], [6] in the framework of classical or axiomatic potential theory, the new topology τ being here the fine topology of Cartan [9], that is, the coarsest topology on X such that every function in the cone $\mathscr{U}$ of all superharmonic functions $\geqq 0$ on X becomes continuous. The space X is now a "harmonic space", satisfying the group of axioms (A_1) in Brelot's axiomatic theory of harmonic functions [4]. It was proved by Brelot [5], [6] that[3)]

$$\hat{R}_{\inf_{\alpha} f_{\alpha}} = \widehat{\inf_{\alpha}} \hat{R}_{f_{\alpha}},$$

$$\hat{R}_{\sup_{\alpha} f_{\alpha}} = \sup_{\alpha} \hat{R}_{f_{\alpha}},$$

or in the equivalent integrated form

$$\int \hat{R}_{\inf_{\alpha} f_{\alpha}} dm = \inf_{\alpha} \int \hat{R}_{f_{\alpha}} dm,$$

$$\int \hat{R}_{\sup_{\alpha} f_{\alpha}} dm = \sup_{\alpha} \int \hat{R}_{f_{\alpha}} dm.$$

Here (f_{α}) denotes an upward (resp. downward) directed family of finely l.s.c. (resp. finely u.s.c.) functions $\geqq 0$ on the harmonic space X. In the second case the finely u.s.c. functions f_{α} should, moreover, be majorized by a *semibounded potential* V.[4)] The measure m may be any harmonic

[3)] For any function $f \in \mathscr{F}^+(X)$, $\hat{R}_f$ denotes the infimum of $\{u \in \mathscr{U}_{\infty} | u \geq f\}$ in the complete lattice $\mathscr{U}_{\infty} = \mathscr{U} \cup \{+\infty\}$ of all hyperharmonic functions ≥ 0 on X. Explicitly, the infimum $\widehat{\inf_{\alpha}} u_{\alpha}$ of any family of hyperharmonic functions $u_{\alpha} \in \mathscr{U}_{\infty}$ in this lattice $\mathscr{U}_{\infty}$ is the l.s.c. envelope of the pointwise infimum $\inf_{\alpha} u_{\alpha}$ (from which it differs only in some polar set, that is, a set contained in $\{x \in X | u(x) = +\infty\}$ for some $u \in \mathscr{U}$). In particular, $\hat{R}_f$ is the l.s.c. envelope of the pointwise infimum R_f of $\{u \in \mathscr{U}_{\infty} | u \geq f\}$.

[4)] It is known that any finite valued potential is semibounded. In the classical case of a Green space X with the Green kernel G, a potential $V = G\lambda$ of a measure $\lambda \geq 0$ on X is semibounded if and only if λ does not charge the polar sets and $G\lambda \not\equiv +\infty$.

measure, or more generally any positive Radon measure ($\neq 0$) such that $\int u dm < +\infty$ for every $u \in \mathscr{U}$.

These results of Brelot may be viewed as a particular case of Theorem 3.7, corresponding to the locally finite, upper capacity C defined by

$$C(f) = \int \hat{R}_f dm, \qquad f \in \mathscr{F}^+(X),$$

with m as specified above. This capacity C is known to be sequentially order continuous from below (in case (A_1) of Brelot's axiomatic theory). According to Brelot [7] the fine topology on X is compatible with the quasi topology determined by the set function $A \to \int \hat{R}_1^A dm$ associated with C (see also [17, § 5.6]). Hence it is easily shown by application of Theorem 2.5 that a finely (hence quasi) u.s.c. function $f \geqq 0$ on X is of class $\mathscr{H}_0^*$ (with respect to the above capacity C) if and only if f is majorized quasi everywhere by some hyperharmonic function V of class $\mathscr{H}_0^*$ (the smallest such V being $\hat{R}_f$). And in view of the corollary in § 2.5, a hyperharmonic function $V \geqq 0$ is of class $\mathscr{H}_0^*$ if and only if V is a potential which is semi-bounded in the sense of Brelot [6], or equivalently in the sense of Def. 2.5 above (with the capacity $f \mapsto C(f) = \int \hat{R}_f dm$).

3.9. *σ-finite sets.* A set $A \subset X$ is called *σ-finite* with respect to a capacity $C : \mathscr{F}^+(X) \to [0, +\infty]$ if A can be covered by a sequence of sets A_n such that $C(A_n) < +\infty$. When C is an upper capacity the sets A_n may of course be taken as open sets.

Lemma. *A set $A \subset X$ is σ-finite with respect to an upper capacity C if and only if there exists a function $g \in \mathscr{G}$ such that $C(g) < +\infty$ and $g(x) > 0$ for all $x \in A$.*

Proof. If g has these properties then A is covered by the sets $A_n := \{x \in X | g(x) > 1/n\}$, and $C(A_n) \leqq nC(g) < +\infty$. Conversely, any covering (A_n) as stated gives rise to a function $f := \Sigma 2^{-n} C(A_n)^{-1} 1_{A_n} \in \mathscr{F}^+(X)$ such that $C(f) < +\infty$ and $f > 0$ in A. Since C is an upper capacity there exists $g \in \mathscr{G}$, $g \geqq f$ (> 0 in A) such that $C(g) < +\infty$. ▌

3.10. *Use of a weight function.* Let $C : \mathscr{F}^+(X) \to [0, +\infty]$ denote a capacity in the sense of Def. 1.1, and let a function $f \in \mathscr{F}^+(X)$ be given. The set function $C_f : \mathscr{P}(X) \to [0, +\infty]$ defined by

$$C_f(A) = C(f \cdot 1_A)$$

is then a capacity in the sense of [17] (cf. § 1.2 above for the particular case $f = 1$).

Theorem. *Suppose that the functional $C : \mathscr{F}^+(X) \to [0, +\infty]$ is an upper capacity on a topological space X, that the weight function f is of class $\mathscr{H}^*$, and that the set $X_0 := \{x \in X \mid f(x) = 0\}$ is σ-finite. Then the above set function C_f is an outer capacity:*

$$C_f(A) = \inf\{C_f(G) \mid G \text{ open}, \ G \supset A\}$$

for every set $A \subset X$.

In establishing this latter relation for a specified set A, it suffices to assume that $A \cap X_0$, rather than all of X_0, be σ-finite. Note that, for $f = 1$, the theorem was obtained in § 3.1. As to the notion of outer capacity in general see [17, § 1.5].

Proof. We may suppose that $C_f(A) < +\infty$. Consider functions $h \in \mathscr{H}$ and $g_1, g_2 \in \mathscr{G}$ with $C(g_2) < +\infty$, such that

$$h \leqq f, \qquad g_1 \geqq f \cdot 1_A,$$
$$g_2 > 0 \text{ on } \{x \in A \mid f(x) = 0\}$$

(cf. Lemma 3.9). Let $\varepsilon > 0$, and write

$$g := (1 + \varepsilon)g_1 + \varepsilon g_2.$$

Then $g \in \mathscr{G}$, $g > f$ on A, and hence the set

$$G := \{x \in X \mid g(x) > h(x)\}$$

is open and contains A. Since $g > h \cdot 1_G$, we obtain

$$\begin{aligned} C_f(G) = C(f \cdot 1_G) &\leqq C(h \cdot 1_G) + C(f - h) \\ &\leqq C(g) + C(f - h) \\ &\leqq (1 + \varepsilon)C(g_1) + \varepsilon C(g_2) + C(f - h), \end{aligned}$$

which may be taken as close to $C(f \cdot 1_A) = C_f(A)$ as we please by appropriate choice of h, g_1, and $\varepsilon > 0$. ▌

Remarks. 1) Note that X_0 is σ-finite if $f > 0$, or if C is locally finite and X is of class $\mathscr{K}_\sigma$. Simple examples show that the σ-finiteness hypothesis in the above theorem cannot be dropped. Also the hypothesis $f \in \mathscr{H}^*$ cannot be replaced by f quasi u.s.c. and quasi bounded (cf. § 2.5).

2) Under the same hypotheses as in the above theorem the *functional* C_f defined by $C_f(\varphi) = C(f\varphi)$, $\varphi \in \mathscr{F}^+(X)$, is an upper capacity. (The proof is similar).

CHAPTER II

Capacity as a Sublinear Functional on $\mathscr{C}_0^+$

Throughout this chapter, X denotes a *locally compact* (Hausdorff) space.

4. Extension to a Lower and an Upper Capacity

4.1. **Definition.** *By a capacity on a locally compact space X we understand, in this chapter, an increasing, sublinear functional c defined on $\mathscr{C}_0^+ = \mathscr{C}_0^+(X)$* (the cone of continuous functions on X to $[0, +\infty[$ of compact support) *and with finite values* $\geqq 0$.

Thus we should have, for $\varphi, \varphi_1, \varphi_2 \in \mathscr{C}_0^+$ and $a \in [0, +\infty[$, the following properties

$$(c_1) \qquad [\varphi_1 \leqq \varphi_2] \Rightarrow [c(\varphi_1) \leqq c(\varphi_2)],$$

$$(c_2) \qquad c(a\varphi) = ac(\varphi),$$

$$(c_3) \qquad c(\varphi_1 + \varphi_2) \leqq c(\varphi_1) + c(\varphi_2),$$

and furthermore $0 \leqq c(\varphi) < +\infty$. Note that $c(0) = 0$ on account of (c_2).

4.2. *Extension to $\mathscr{H}_0$ and $\mathscr{G}$.* Given a capacity c on X, we define for $h \in \mathscr{H}_0$ and $g \in \mathscr{G}$

$$c(h) = \inf\{c(\varphi) \mid \varphi \in \mathscr{C}_0^+, \varphi \geqq h\},$$

$$c(g) = \sup\{c(\varphi) \mid \varphi \in \mathscr{C}_0^+, \varphi \leqq g\}.$$

This is permissible and leads to a well-defined extension of c to $\mathscr{G} \cup \mathscr{H}_0$ because $\mathscr{G} \cap \mathscr{H}_0 = \mathscr{C}_0^+$, and c is increasing on $\mathscr{C}_0^+$ by (c_1). Note that $c(h) < +\infty$ for every $h \in \mathscr{H}_0$. This extension of c to $\mathscr{G} \cup \mathscr{H}_0$ is likewise *increasing*, the only non-trivial case being the implication

$$[h \in \mathscr{H}_0, g \in \mathscr{G}, h \leqq g] \Rightarrow [c(h) \leqq c(g)], \tag{1}$$

which is an immediate consequence of the "between theorem" in the form given in Lemma 3.4.

4.3. **Theorem.** (a) *For any upward directed family of functions $g_\alpha \in \mathscr{G}$*

$$c(\sup_\alpha g_\alpha) = \sup_\alpha c(g_\alpha).$$

(b) *For any downward directed family of functions $h_\alpha \in \mathscr{H}_0$*

$$c(\inf_\alpha h_\alpha) = \inf_\alpha c(h_\alpha).$$

Proof. (a) Write $g := \sup_{\alpha} g_{\alpha}$. Since $g \in \mathscr{G}$, there exists for any number $t < c(g)$ a function $\varphi \in \mathscr{C}_0^+$ such that $\varphi \leqq g$ and $c(\varphi) > t$. Choose $\psi \in \mathscr{C}_0^+$ so that $\psi = 1$ on the compact support of φ, and $\psi \leqq 1$ everywhere. Since the downward directed family of functions $(\varphi - g_{\alpha})^+ \in \mathscr{H}_0$ converges pointwise to 0, the convergence is uniform by Dini's theorem. Denoting by β a fixed index, there exists for any $\varepsilon > 0$ an index $\alpha \succcurlyeq \beta$ such that $(\varphi - g_{\alpha})^+ < \varepsilon$ everywhere. It follows by our choice of ψ that $\varphi \leqq g_a + \varepsilon\psi$ everywhere. From this inequality we obtain[5]

$$c(\varphi) \leqq c(g_{\alpha}) + \varepsilon c(\psi).$$

Choosing $\varepsilon > 0$ small enough so that $\varepsilon c(\psi) \leqq c(\varphi) - t$, we have now established the existence of an index α such that $c(g_{\alpha}) \geqq t$, and we have thus obtained the non-trivial inequality $c(g) \leqq \sup c(g_{\alpha})$.

The case (b) is analogous and even simpler, cf. also the similar proof of Theorem 3.6 (a). Actually (b) follows from Theorems 3.6 (a) and 4.5. ▌

4.4. *The lower capacity c_* and the upper capacity c^*.* These are defined for arbitrary $f \in \mathscr{F}^+(X)$ by

$$c_*(f) = \sup\{c(h) \,|\, h \in \mathscr{H}_0,\ h \leqq f\},$$

$$c_*(f) = \inf\{c(g) \,|\, g \in \mathscr{G},\ g \geqq f\}.$$

The functionals $c_*, c^* : \mathscr{F}^+(X) \to [0, +\infty]$ are evidently *increasing*, and we get from (1), § 4.2,

$$c_*(f) \leqq c^*(f) \tag{2}$$

for every $f \in \mathscr{F}^+(X)$. A function $f \in \mathscr{F}^+(X)$ such that $c_*(f) = c^*(f)$ is called *capacitable* (with respect to the capacity c), or c-capacitable. For any capacitable function f we shall allow ourselves to write simply $c(f)$ in place of $c_*(f)$ or $c^*(f)$, and to call $c(f)$ the capacity of f. This is permissible (and leads to our ultimate extension of the original functional c on $\mathscr{C}^+$) on account of the following lemma.

Lemma. *Every function f of class $\mathscr{G}$ or $\mathscr{H}_0$ is capacitable, and*

$$c_*(f) = c^*(f) = c(f).$$

Proof. If $f \in \mathscr{H}_0$ we have

$$c^*(f) \leqq c(f) = c_*(f).$$

[5]) In fact, $(\varphi - \varepsilon\psi)^+ \in \mathscr{C}_0^+$, $(\varphi - \varepsilon\psi)^+ \leq g_{\alpha}$, and hence $c(\varphi) \leq c((\varphi - \varepsilon\psi)^+) + c(\varepsilon\psi) \leq c(g_{\alpha}) + \varepsilon c(\psi)$.

Here the inequality follows from the definitions of $c^*(f)$ and of $c(f)$ (for $f \in \mathscr{H}_0$) because $\mathscr{G} \supset \mathscr{C}_0^+$.—The case $f \in \mathscr{G}$ is quite analogous. ▌

4.5. *Further properties of* c_* *and* c^*. It is immediately verified that c_* and c^* are *positive homogeneous* (just like c itself on $\mathscr{C}_0^+$), that is,

$$c_*(af) = ac_*(f), \; c^*(af) = ac^*(f)$$

for $f \in \mathscr{F}^+(X)$, $0 \leqq a < +\infty$. Moreover, c^* is *countably subadditive*:

$$c^*\left(\sum_{n \in \boldsymbol{N}} f_n\right) \leqq \sum_{n \in \boldsymbol{N}} c^*(f_n) \qquad (f_n \in \mathscr{F}^+(X)). \tag{3}$$

The proof of this is easily reduced to the case $f_n \in \mathscr{G}$, in which case Theorem 4.3 (a) allows us to reduce further to the case of a finite sum, or, by recurrence, to a sum of just 2 functions of class $\mathscr{G}$. Approximating each of these two functions by the upward directed family of all its minorants of class $\mathscr{C}_0^+$, and applying Theorem 4.3 (a) once again, we have finally reduced (3) to the subadditivity of c on $\mathscr{C}_0^+$ as stipulated in (c_3) of Def. 4.1.

Next we propose to show that

$$c_*(f_1 + f_2) \leqq c_*(f_1) + c^*(f_2) \tag{4}$$

for all $f_1, f_2 \in \mathscr{F}^+(X)$. Let $h \in \mathscr{H}_0$, $h \leqq f_1 + f_2$, $g \in \mathscr{G}$, $g \geqq f_2$. Then $(h-g)^+ \in \mathscr{H}_0$, $(h-g)^+ \leqq f_1$, and hence, by (3) and Lemma 4.4,

$$c(h) \leqq c((h-g)^+) + c(g) \leqq c_*(f_1) + c(g).$$

This establishes (4) because $c(h)$ and $c(g)$ may be taken as close as we please to $c_*(f_1 + f_2)$ and $c^*(f_2)$, respectively.

In view of (3) we have established, in particular, the following theorem serving to justify our use of the name upper capacity for c^*.

Theorem. *For any capacity* $c : \mathscr{C}_0^+(X) \to [0, +\infty[$ *(in the sense of Def. 4.1) on a locally compact space* X *the associated upper capacity* $c^* : \mathscr{F}^+(X) \to [0, +\infty]$ *is an upper capacity in the sense of Def.* 3.1. *Moreover,* c^* *is locally finite (Def. 3.5). A function* $f \in \mathscr{F}^+(X)$ *is c-capacitable* ($c^*(f) = c_*(f)$) *if and only if* f *is* $(c^*, \mathscr{H}_0)$*-capacitable in the sense of § 1.6.*

Remarks. 1) In order that a locally finite, upper capacity $C : \mathscr{F}^+(X) \to [0, +\infty]$ (in the sense of Def. 3.1) on the locally compact space X have the form $C = c^*$ for some c as above, it is necessary and sufficient that

$$C(g) = \sup\{C(\varphi) \,|\, \varphi \in \mathscr{C}_0^+, \; \varphi \leqq g\} \tag{5}$$

for every $g \in \mathscr{G}$. In the affirmative case c is of course uniquely determined as the restriction of C to $\mathscr{C}_0^+$. Note also that, according to the between theorem (in the form given in Lemma 3.4 above) it suffices to verify the apparently weaker condition derived from (5) by replacing $\mathscr{C}_0^+$ by $\mathscr{H}_0$ (in other words the $(C,\mathscr{H}_0)$-capacitability of any $g \in \mathscr{G}$).

If X has a countable base, and if C is sequentially order continuous from below, then (5) holds because any $g \in \mathscr{G}$ is representable as the pointwise supremum of an increasing sequence of functions of class $\mathscr{C}_0^+$.

2) If c^* is sequentially order continuous from below (on $\mathscr{F}^+(X)$), it follows from Choquet's theory that every $\mathscr{H}_0$-Souslin function is c-capacitable in view of Theorem 1.6 above because c^* is order continuous from above on $\mathscr{H}_0$ according to Theorem 3.6 (a) or Theorem 4.3 (b).

3) In view of the above theorem we may of course freely use the concepts and results from Chapter I, and also from [17], taking $C = c^*$, the upper capacity associated with a given capacity $c : \mathscr{C}_0^+(X) \to [0, +\infty[$. In particular, we have the closed classes $\mathscr{G}^*$, $\mathscr{H}^*$, $\mathscr{H}_0^*$, etc. The expression *quasi everywhere* (q.e.) means: everywhere except in some set E such that $c^*(E) = 0$. (We put $c^*(E) = c^*(1_E)$ for every set $E \subset X$, cf. § 4.7.)

4.6. **Lemma.** *The capacitable functions $f \in \mathscr{F}^+(X)$ form a closed subset of $\mathscr{F}^+(X)$ in the c^*-metric topology. In particular, every function of class $\mathscr{G}^*$ or $\mathscr{H}_0^*$ is capacitable, in the latter case with finite capacity.*

Proof. Let $f \in \mathscr{F}^+(X)$, and suppose that there corresponds to any $\varepsilon > 0$ a capacitable function $\varphi \in \mathscr{F}^+(X)$ such that $c^*(|f-\varphi|) < \varepsilon$. Since $f \leqq \varphi + (f-\varphi)^+$, and vice versa, we obtain from (3) and (4), § 4.5,

$$c^*(f) \leqq c^*(\varphi) + c^*((f-\varphi)^+) \leqq c_*(\varphi) + \varepsilon,$$

$$c_*(\varphi) \leqq c_*(f) + c^*((\varphi - f)^+) \leqq c_*(f) + \varepsilon,$$

and hence $c^*(f) \leqq c_*(f) + 2\varepsilon$. The last assertion follows now from the finiteness of c on $\mathscr{H}_0$. ∎

4.7. *Capacity with respect to a weight function.* Let $c : \mathscr{C}_0^+(X) \to [0, +\infty[$ denote a capacity in the sense of Def. 4.1 on a locally compact space X. For any compact set $K \subset X$ we put

$$c(K) := c(1_K).$$

More generally, let a function $f \in \mathscr{H}^*$ be given, and define for any compact set K

$$c_f(K) := c(f \cdot 1_K).$$

This makes sense because $f \cdot 1_K \in \mathscr{H}_0^*$ by virtue of Theorem 2.6 (iv), and hence is c-capacitable by the above lemma. Clearly c_f is finite valued, increasing, and subadditive, and $c_f(\emptyset) = 0$.

It will be shown in the theorem below that the set function c_f, defined on the class $\mathscr{K} = \mathscr{K}(X)$ of all compact subsets of X, is "continuous from the right", and hence is a capacity in the sense of Choquet [10, § 15]. We call this set function c_f the set function, or capacity, with the weight function $f \in \mathscr{H}^*$ associated with c. In the case $f = 1$, where we write $c_1(K) = c(K)$, we simply speak of the associated set function.

From the finite and increasing set function $c_f : \mathscr{K}(X) \to [0, +\infty]$ (where $f \in \mathscr{H}^*$) we derive in the usual way inner and outer set functions $c_{f*}, c_f^* : \mathscr{P}(X) \to [0, +\infty]$:

$$c_{f*}(A) := \sup\{c_f(K) \mid K \text{ compact}, \quad K \subset A\},$$
$$c_f^*(A) := \inf\{c_{f*}(G) \mid G \text{ open}, \qquad G \supset A\}.$$

Clearly these set functions are increasing, and take the value 0 at the void set $\emptyset$. Moreover $c_{f*}(A) \leqq c_f^*(A)$ for any set $A \subset X$. We call a set A capacitable with respect to c_f, or c_f-capacitable, if $c_{f*}(A) = c_f^*(A)$. In that case we may write simply $c_f(A)$ for the common value. This is justified because compact sets are c_f-capacitable in view of the following theorem. (It is trivial that open sets are c_f-capacitable.)

Theorem. *For any $f \in \mathscr{H}^*$ and any set $A \subset X$ we have $c_{f*}(A) = c_*(f \cdot 1_A)$. If moreover $\{x \in A \mid f(x) = 0\}$ is σ-finite with respect to c^* then $c_f^*(A) \leqq c^*(f \cdot 1_A)$. The sign of equality subsists here if, in addition, $f \in \mathscr{G}^*$.*

Proof. Ad $c_{f*}(A)$. For any compact set $K \subset A$ we have $f \cdot 1_K \in \mathscr{H}_0^*$, $f \cdot 1_K \leqq f \cdot 1_A$, and hence

$$c_f(K) := c(f \cdot 1_K) \leqq c_*(f \cdot 1_A).$$

This shows that $c_{f*}(A) \leqq c_*(f \cdot 1_A)$. Conversely, let $h \in \mathscr{H}_0$, $h \leqq f \cdot 1_A$, and write

$$K_n := \{x \in X \mid h(x) \geqq 1/n\}$$

for $n \in \boldsymbol{N}$. Then K_n is a compact subset of A, and so $c_f(K_n) \leqq c_{f*}(A)$. Denoting by K the compact support of h, we have

$$h \leqq n^{-1} 1_K + f \cdot 1_{K_n},$$

and hence from the sublinearity of c^*

$$c(h) \leqq n^{-1} c(1_K) + c_f(K_n).$$

Since $c(1_K) < +\infty$ and $c_f(K_n) \leqq c_{f*}(A)$, we conclude for $n \to +\infty$ that $c(h) \leqq c_{f*}(A)$, and consequently $c_*(f\cdot 1_A) \leqq c_{f*}(A)$.

Ad $c_f^*(A)$. According to Theorem 3.10,

$$c^*(f\cdot 1_A) = \inf\{c^*(f\cdot 1_G) \mid G \text{ open, } G \supset A\},$$

and hence $c^*(f\cdot 1_A) \geqq c_f^*(A)$ because $c^*(f\cdot 1_G) \geqq c_*(f\cdot 1_G) = c_{f*}(G)$. This argument shows, moreover, that $c^*(f\cdot 1_A) = c_f^*(A)$ holds provided that $c^*(f\cdot 1_G) = c_*(f\cdot 1_G)$, that is, if $f\cdot 1_G$ is c-capacitable for every open set G. And this is the case, in particular, if $g \in \mathscr{G}^*$, for then $g\cdot 1_G \in \mathscr{G}^*$ for every open set G. (For another case where $f\cdot 1_G$ is capacitable for all open sets G see Remark 1 below.) ▌

Corollary. *Let $f \in \mathscr{H}^*$. The inner capacity c_{f*} with the weight function f is countably subadditive on universally measurable sets. The outer capacity c_f^* is countably subadditive.*

The former statement follows from Theorem 7.1 below in view of the identity $c_{f*}(A) = c_*(f\cdot 1_A)$. The latter statement follows easily from the former applied to open sets.

Remarks. 1) Suppose that the upper capacity c^* is *sequentially order continuous from below* (on arbitrary functions $X \to [0, +\infty]$). Suppose further that *either* (i) $f \in \mathscr{G}^* \cap \mathscr{H}^*$, and $\{x \in X \mid f(x) = 0\}$ is σ-finite; *or* (ii) $f \in \mathscr{H}^*$, and X has a countable base. Then

$$c_f^*(A) = c^*(f\cdot 1_A), \qquad c_{f*}(A) = c_*(f\cdot 1_A)$$

for arbitrary sets $A \subset X$. It follows that the outer capacity c_f^* with respect to the weight function f is sequentially order continuous from below on arbitrary sets, and hence, by Choquet's theory, that any K-analytic set $A \subset X$ is c_f-capacitable (Choquet [10, § 30], see also Sion [19]).

As to the proof of the equality $c_f^*(A) = c^*(f\cdot 1_A)$ in the remaining case (ii) we merely have to note (cf. the end of the proof of Theorem 4.7) that $f\cdot 1_G$ is c-capacitable for every open set G; and this is clear also in case (ii) since $f\cdot 1_G$ is then equivalent to a function of class $(\mathscr{H}_0)_\sigma$ and hence c-capacitable because c^* is sequentially order continuous from below. (In fact, the open set G is of class $\mathscr{K}_\sigma$ here, and $f \in \mathscr{H}^*$ is obviously equivalent to a function of class $\mathscr{H}_\sigma$.)

2) Consider a function $f \in \mathscr{G}^* \cap \mathscr{H}^*$ such that $\{x \in X \mid f(x) = 0\}$ is σ-finite with respect to c^*. For any $\varphi \in \mathscr{C}_0^+$ we have $f\varphi \in \mathscr{H}_0^*$ by Theorem 2.6. Hence a new capacity c_f (in the sense of Def. 4.1) is defined by

$$c_f(\varphi) := c(f\varphi) \qquad (\varphi \in \mathscr{C}_0^+).$$

By the same method as in the proof of the above theorem it can be shown that the associated lower and upper capacities are given by

$$c_{f*}(\varphi) = c_*(f\varphi), \qquad \overset{*}{c_f}(\varphi) = c^*(f\varphi) \qquad (\varphi \in \mathscr{F}^+(X)).$$

4.8. Let us now return to the principal case of the weight function $f = 1$ and the set function $c : \mathscr{K} \to [0, +\infty[$ associated with the given capacity functional $c : \mathscr{C}_0^+ \to [0, +\infty[$ by the definition $c(K) = c(1_K)$. This set function is finite valued, increasing, subadditive, continuous from the right (in the sense of CHOQUET [10, § 15], that is, $c^*(K) = c(K)$ for all $K \in \mathscr{K}$), and $c(\emptyset) = 0$. Not every set function $c : \mathscr{K} \to [0, +\infty[$ with these properties is associated in this way with a capacity in the sense of Def. 4.1. As observed by CHOQUET [10, § 53.7] a simple necessary condition on the set function c is that

$$2c(A \cup B \cup C) \leqq c(A \cup B) + c(B \cup C) + c(C \cup A)$$

for arbitrary compact sets A, B, C, and this condition is not always fulfilled.[6] It seems difficult to obtain a simple necessary and sufficient condition. As shown by CHOQUET [10, § 54.2] it is *sufficient* that c be *strongly* subadditive in the sense that

$$c(A \cup B) + c(A \cap B) \leqq c(A) + c(B)$$

for arbitrary compact sets A, B. This condition, however, is not a necessary one, as it appears say from the usual capacity associated with a kernel (cf. e.g. [16]). And if c is strongly subadditive (on compact sets) there may exist several extensions to a functional capacity in the sense of Def. 4.1.[7]

5. Representation of a Capacity by a Set of Measures

We show (Theorem 5.3) that every capacity $c : \mathscr{C}_0^+(X) \to [0, +\infty[$ may be obtained as the supremum of a family $\mathscr{S}$ of linear capacities (that is, positive Radon measures) $\mu : \mathscr{C}_0^+(X) \to [0, +\infty[$,

$$c(f) = \sup_{\mu \in \mathscr{S}} \mu(f), \qquad f \in \mathscr{C}_0^+.$$

The largest such family is the set $\mathscr{S}_c$ of all positive Radon measures μ such that $\mu \leqq c$ (that is, $\mu(f) \leqq c(f)$ for all $f \in \mathscr{C}_0^+$). Conversely, for any vaguely

[6]) Example: Let X consist of 3 points; let $c(K) = 1$ for any set $K \subset X$ consisting of 1 or 2 points; and let $c(X) = 2$ (and $c(\emptyset) = 0$).

[7]) Take for c the newtonian capacity c_1 (as a functional on $\mathscr{C}_0^+$, see § 5.7 below). The restriction of c_1 to (indicator functions for) compact sets is then strongly subadditive as shown by CHOQUET [10, ch. 2], but the extension of this latter set function constructed in CHOQUET [10, § 54.2] is not equal to the functional c_1.

bounded set $\mathscr{S}$ of positive Radon measures the above supremum clearly defines a capacity c. We show (Theorem 5.4) that $\mathscr{S}_c$ is the hereditary convex closure of $\mathscr{S}$.

5.1. *The strong topology on* $\mathscr{C}_0$. The vector space $\mathscr{C}_0 = \mathscr{C}_0(X)$ of all finite real valued continuous functions of compact support on the locally compact space X has a well known locally convex separated topology called the *strong* topology. If X is compact, the strong topology on $\mathscr{C}_0(X) = \mathscr{C}(X)$ is simply the uniform topology defined by means of the uniform norm $f \mapsto \max_{x \in X} |f(x)|$, $f \in \mathscr{C}_0(X)$. If X is locally compact, but not compact, the strong topology on $\mathscr{C}_0(X)$ is defined as the *inductive limit* of the uniform topologies on the subspaces $\mathscr{C}_0(X,K)$, where K ranges over all compact subsets of X. Here $\mathscr{C}_0(X,K)$ denotes the set of all functions $f \in \mathscr{C}_0(X)$ vanishing outside K. The topology on $\mathscr{C}_0^+(X)$ induced by the strong topology on $\mathscr{C}_0(X)$ is called the strong topology on $\mathscr{C}_0^+(X)$. We refer to BOURBAKI [3, ch. II, § 4, no. 4].

Theorem. *Any capacity* $c : \mathscr{C}_0^+(X) \to [0, +\infty[$ *on a locally compact space* X *is continuous in the strong topology on* $\mathscr{C}_0^+(X)$.

Proof. We extend c from $\mathscr{C}_0^+$ to $\mathscr{C}_0$ by defining

$$\tilde{c}(f) := c(|f|) \qquad (f \in \mathscr{C}_0). \tag{6}$$

Clearly $\tilde{c}$ is homogeneous, that is,

$$\tilde{c}(af) = |a|\tilde{c}(f) \tag{7}$$

for any real number a and any $f \in \mathscr{C}_0$. Since c is subadditive and increasing on $\mathscr{C}_0^+$, we have

$$\tilde{c}(f_1 + f_2) \leqq \tilde{c}(f_1) + \tilde{c}(f_2), \tag{8}$$

$$|f_1| \leqq |f_2| \;\Rightarrow\; \tilde{c}(f_1) \leqq \tilde{c}(f_2) \tag{9}$$

for all $f_1, f_2 \in \mathscr{C}_0$. The properties (7), (8) together with the finiteness of $\tilde{c}$ amount to saying that $\tilde{c}$ is a *seminorm* on $\mathscr{C}_0$. In particular, for $f_1, f_2 \in \mathscr{C}_0$,

$$|\tilde{c}(f_1) - \tilde{c}(f_2)| \leqq \tilde{c}(f_1 - f_2). \tag{10}$$

For functions $f \in \mathscr{C}_0(X,K)$ we have $|f| \leqq a \cdot 1_K$ with $a := \max_{x \in K} |f(x)|$, and hence

$$\tilde{c}(f) = c(|f|) \leqq a \cdot c(K).$$

Applying this to $f = f_1 - f_2$ with $f_1, f_2 \in \mathscr{C}_0(X,K)$, we obtain from (10)

$$|\tilde{c}(f_1) - \tilde{c}(f_2)| \leqq c(K) \max_{x \in K} |f_1(x) - f_2(x)|.$$

This inequality shows that c is indeed continuous relative to each $\mathscr{C}_0(X,K)$ with the uniform topology, and hence continuous on all of $\mathscr{C}_0(X)$ with the strong topology according to Bourbaki [3, ch. II, § 4, prop. 5] because $\tilde{c}$ is a seminorm. ▎

5.2. *Measures and integration.* We give a brief exposition of the theory of Radon measures (integrals) on a locally compact space (cf. Bourbaki [2], Cartan [8]), noting that such a measure may be identified with an *additive capacity*.

By a (real valued Radon) measure μ on a locally compact space X is understood a (strongly) continuous real linear form μ on $\mathscr{C}_0(X)$. The vector space $\mathscr{M} = \mathscr{M}(X)$ of all measures on X is thus the dual space of $\mathscr{C}_0$. The value of a measure $\mu \in \mathscr{M}$ at a function $f \in \mathscr{C}_0$ is denoted by

$$\mu(f) = \int f d\mu.$$

The weak*-topology on the dual space $\mathscr{M}$ is called the *vague* topology on $\mathscr{M}$. A set $\mathscr{S} \subset \mathscr{M}$ is relatively compact (in the vague topology) if and only if $\mathscr{S}$ is vaguely bounded in the sense that the linear form $\mu \mapsto \mu(f)$ is bounded on $\mathscr{S}$ for every fixed $f \in \mathscr{C}_0$.

We shall mainly consider *positive* measures μ, that is, measures such that $\mu(f) \geqq 0$ for every $f \in \mathscr{C}_0^+$. The set of all positive measures on X is a convex cone denoted by $\mathscr{M}^+ = \mathscr{M}^+(X)$. The restriction of a positive measure μ to $\mathscr{C}_0^+$ is a *capacity* on X in the sense of Def. 4.1, but with the further property of being *additive* (not just subadditive). Conversely, it is well known that any additive capacity c on X has a unique extension to a positive Radon measure μ on X, viz.

$$\mu(f) = \mu(f^+) - \mu(f^-).$$

Having thus identified a positive measure μ (or rather its restriction to $\mathscr{C}_0^+$) with an additive capacity, we note that the integral $\int h d\mu = \mu(h)$ of a function $h \in \mathscr{H}_0$ and the integral (Bourbaki: upper integral) $\int g d\mu = \mu(g)$ of a function $g \in \mathscr{G}$[8]) coincide by definition with the capacity of h and g, respectively, as defined in § 4.2.

Furthermore the *lower* and the *upper* integral of an arbitrary function $f \in \mathscr{F}^+(X)$ with respect to μ are precisely the lower and the upper capacity of f, respectively, as defined in § 4.4:

$$\int_* f d\mu = \mu_*(f) = \sup\{\int h d\mu \,|\, h \in \mathscr{H}_0,\ h \leqq f\},$$
$$\int^* f d\mu = \mu^*(f) = \inf\{\int g d\mu \,|\, g \in \mathscr{G},\ g \geqq f\}.$$

[8]) See § 2 for the classes $\mathscr{H}_0$ and $\mathscr{G}$.

Conforming with our general convention we shall allow ourselves to write simply $\int f d\mu = \mu(f)$ for these two integrals whenever they coincide, that is, when f is *capacitable* with respect to the measure μ.

It is well known that the *upper* integral μ^* is sequentially order continuous from below:[9)]

$$[f_n \nearrow f,\ f_n \in \mathscr{F}^+(X)] \Rightarrow [\mu^*(f_n) \to \mu^*(f)]. \tag{11}$$

Moreover μ^* is countably subadditive (special case of (3), § 4.5).

The *lower* integral μ_* is known to possess similar properties when considered only on *μ-measurable* functions. Thus

$$[f_n \nearrow f] \Rightarrow [\mu_*(f_n) \to \mu_*(f)], \tag{12}$$

$$\mu_*\Big(\sum_{n \in \boldsymbol{N}} f_n\Big) \leqq \sum_{n \in \boldsymbol{N}} \mu_*(f_n), \tag{13}$$

for any sequence of μ-measurable functions $f_n \in \mathscr{F}^+(X)$.

Recall also that a μ-measurable function $f \in \mathscr{F}^+(X)$ is *μ-capacitable* provided that $\{x \in X \mid f(x) > 0\}$ is σ-finite with respect to μ. Finally, a function $f \in \mathscr{F}^+(X)$ is *μ-integrable* if and only if $\mu^*(f) = \mu_*(f) < +\infty$.

The quasi topological notions discussed in [17], and the classes $\mathscr{G}^*$, $\mathscr{H}^*$, $\mathscr{H}_0^*$ introduced in § 2 reduce to well known concepts in the present case of the upper capacity μ^* (the upper integral) associated with a positive measure μ. Thus $\mathscr{H}_0^*$ consists of all non-negative μ-integrable functions. If the space X is countable at infinity, quasi continuity (or quasi semicontinuity) with respect to μ^* reduces to measurability with respect to μ. In particular, $\mathscr{G}^*$ is then the class of all non-negative μ-measurable functions, and $\mathscr{H}^*$ the class of all non-negative locally μ-integrable functions (cf. Theorems 2.5 and 2.6).

The *trace* μ_A of a measure $\mu \in \mathscr{M}^+$ on a μ-measurable set $A \subset X$ is defined by

$$\mu_A(\varphi) = \int \varphi \cdot 1_A d\mu \qquad (\varphi \in \mathscr{C}_0^+).$$

(Observe that $\varphi \cdot 1_A$ is μ-integrable.) The total mass of μ_A is

$$\mu_A(X) = \mu_*(A).$$

A measure $\mu \in \mathscr{M}^+$ is said to be *concentrated* on (or *carried* by) a set A if $\complement A$ is locally μ-negligible, or equivalently if A is μ-measurable and $\mu_A = \mu$. It follows then that $\mu(X) = \mu_*(A)$. If A is closed, or if e.g. $\mu_*(A) < +\infty$,

9) Not every capacity c has the property that c^* is sequentially order continuous from below, cf. § 5.6 below. On the other hand the capacities encountered in potential theory do have this property under very general circumstances.

then $\mu^*(\complement A) = 0$, that is, $\complement A$ is μ-negligible. A measure μ is, therefore, concentrated on a *closed* set A if and only if $\operatorname{supp}\mu \subset A$. Here $\operatorname{supp}\mu$ denotes the (closed) *support* of μ (the smallest closed set carrying μ). Finally note that, for any measure $\mu \in \mathscr{M}^+$ and any μ-measurable set A, the trace μ_A is concentrated on A.

Definition. *A set $\mathscr{S} \subset \mathscr{M}^+$ of positive measures is called hereditary (from above) if*

$$\forall \mu \in \mathscr{S} \quad \forall \nu \in \mathscr{M}^+ : [\nu \leqq \mu] \Rightarrow [\nu \in \mathscr{S}].$$

5.3. *The representation theorem.* Returning now to the case of an arbitrary capacity c on X in the sense of Def. 4.1, we write

$$\mathscr{S}_c := \{\mu \in \mathscr{M}^+ \mid \mu \leqq c\}, \tag{14}$$

where $\mu \leqq c$ means $\mu(f) \leqq c(f)$ for all $f \in \mathscr{C}_0^+$.

Theorem.[10] *Every capacity $c : \mathscr{C}_0^+(X) \to [0, +\infty[$ is representable as the upper envelope $c = \sup\limits_{\mu \in \mathscr{S}_c} \mu$ of the associated set $\mathscr{S}_c$ of all positive measures $\mu \leqq c$. More precisely, we have for every $f \in \mathscr{C}_0^+$,*

$$c(f) = \max_{\mu \in \mathscr{S}_c} \mu(f).$$

Proof. For every $f_0 \in \mathscr{C}_0^+$ we shall prove the existence of a measure $\mu \in \mathscr{S}_c$ such that $\mu(f_0) = c(f_0)$. According to the Hahn-Banach theorem, applied to the locally convex space $\mathscr{C}_0$ with the strong topology (§ 5.1) and the continuous semi-norm $\tilde{c}$ defined in (6), there exists an extension of the linear form $t f_0 \mapsto t\,\tilde{c}(f_0)$ (on the 1-dimensional subspace generated by f_0) to a continuous linear form λ on $\mathscr{C}_0$ such that $|\lambda(f)| \leqq \tilde{c}(f)$ for all $f \in \mathscr{C}_0$. Thus λ is a (Radon) measure on X. The positive part $\mu := \lambda^+$ of λ has the desired properties. In fact, $\mu \geqq \lambda$, and hence

$$\mu(f_0) \geqq \lambda(f_0) = \tilde{c}(f_0) = c(f_0)$$

because $f_0 \geqq 0$. On the other hand, for every $f \in \mathscr{C}_0^+$,

$$\mu(f) = \sup\{\lambda(h) \mid h \in \mathscr{C}_0^+,\ h \leqq f\} \leqq c(f)$$

since $\lambda(h) \leqq \tilde{c}(h) = c(h) \leqq c(f)$. ▌

The set $\mathscr{S}_c \subset \mathscr{M}^+$ defined in (14) is evidently *hereditary* (cf. end of § 5.2), *convex*, and vaguely *compact*.

[10]) This result is mentioned in Choquet [10, § 53.7].

5.4. On the other hand, for any set $\mathscr{S}$ of positive measures, the functional c defined on $\mathscr{C}_0^+$ as the upper envelope of $\mathscr{S}$,

$$c(f) = \sup_{\mu \in \mathscr{S}} \mu(f) \qquad (f \in \mathscr{C}_0^+) \tag{15}$$

is evidently increasing and sublinear. Thus c is a capacity in the sense of Def. 4.1 if and only if c is finite valued, or equivalently if $\mathscr{S}$ is vaguely *bounded* (that is, vaguely relatively compact).

Theorem. *For any vaguely bounded set* $\mathscr{S} \subset \mathscr{M}^+$ *the upper envelope* $c = \sup_{\mu \in \mathscr{S}} \mu$ *is a capacity. The associated set* $\mathscr{S}_c$ *of all positive measures* $\mu \leqq c$ *is the hereditary convex closure of* $\mathscr{S}$.[11)]

Proof. It remains to establish that the hereditary convex closure $\mathscr{T}$ of $\mathscr{S}$ coincides with $\mathscr{S}_c$. Since $\mathscr{S}_c$ is hereditary, convex, and closed, and since $\mathscr{S}_c \supset \mathscr{S}$, we have $\mathscr{S}_c \supset \mathscr{T}$. To prove that $\mathscr{S}_c \subset \mathscr{T}$ we use the duality between $\mathscr{M}$ and $\mathscr{C}_0$ determined by the bilinear form $(\mu,f) \rightarrow \mu(f) = \int f d\mu$. The polar $\mathscr{T}^0 \subset \mathscr{C}_0$ of $\mathscr{T} (\subset \mathscr{M})$ consists by definition of all $f \in \mathscr{C}_0$ such that

$$\mu(f) \leqq 1 \qquad \text{for every } \mu \in \mathscr{T}. \tag{16}$$

Similarly, the bi-polar $\mathscr{T}^{00} \subset \mathscr{M}$ consists of all $\mu \in \mathscr{M}$ such that

$$\mu(f) \leqq 1 \text{ for every } f \in \mathscr{T}^0. \tag{17}$$

Since $\mathscr{T}$ is convex, closed, and contains 0 (being hereditary), it is known that $\mathscr{T}^{00} = \mathscr{T}$ (Bourbaki [3, ch. II, § 6, th. 1]). In order to prove that $\mathscr{S}_c \subset \mathscr{T}^{00}$ we consider any measure $\mu_0 \in \mathscr{S}_c$ and propose to verify (17) with $\mu = \mu_0$. It suffices to consider positive functions $f \in \mathscr{T}^0$ because $\mathscr{T}$ is hereditary. In fact, for any $f \in \mathscr{T}^0$ we have $f^+ \in \mathscr{T}^0$,[12)] hence $\mu_0(f) \leqq \mu_0(f^+) \leqq 1$ because $f \leqq f^+$ and $\mu_0 \geqq 0$. Consider, therefore, a function $f \geqq 0$ in $\mathscr{T}^0$. Since $\mathscr{T} \supset \mathscr{S}$ we infer from (16) that $\mu(f) \leqq 1$ for every $\mu \in \mathscr{S}$, or equivalently that $c(f) \leqq 1$ according to (15). When $\mu_0 \in \mathscr{S}_c$ we conclude from (14) that $\mu_0(f) \leqq c(f) \leqq 1$, and so $\mu_0 \in \mathscr{T}^{00} = \mathscr{T}$. ∎

Corollary. *A vaguely bounded set of positive measures determines the same enveloping capacity as its hereditary convex closure.*

11) By the hereditary convex closure of a set $\mathscr{S} \subset \mathscr{M}^+$ is meant the smallest hereditary, convex and vaguely closed subset of $\mathscr{M}^+$ containing $\mathscr{S}$.

12) For any $\mu \in \mathscr{T}$ the trace ν of μ on $\{x \in X \mid f(x) > 0\}$ belongs to $\mathscr{T}$ since $\mathscr{T}$ is hereditary. For $f \in \mathscr{T}^0$ we thus obtain $f^+ \in \mathscr{T}^0$ because $\mu(f^+) = \nu(f) \leqq 1$ for every $\mu \in \mathscr{T}$.

5.5. *Representation of the lower capacity.* Let $\mathscr{S}$ denote a vaguely *compact* set of positive measures on the locally compact space X, and let c denote the enveloping capacity as defined in (15).

Theorem. *For any function $h \in \mathscr{H}$ we have*

$$c(h) = \max_{\mu \in \mathscr{S}} \int h d\mu. \tag{18}$$

For any function $f \in \mathscr{F}^+(X)$

$$c_*(f) = \sup_{\mu \in \mathscr{S}} \int_* f d\mu, \tag{19}$$

$$c^*(f) \geqq \sup_{\mu \in \mathscr{S}} \int^* f d\mu. \tag{20}$$

Proof. When $h \in \mathscr{H}_0$, the mapping $\mu \mapsto \int h d\mu$ of $\mathscr{M}^+$ into $[0, +\infty[$ is u.s.c. in the vague topology, being the lower envelope of the family of vaguely continuous mappings $\mu \mapsto \int \varphi d\mu$ as φ ranges over the downward directed family Φ of all functions $\varphi \in \mathscr{C}_0^+$ with $\varphi \geqq h$. Hence this mapping has a greatest value $c'(h)$ on the compact set $\mathscr{S} \subset \mathscr{M}^+$. For any $\mu \in \mathscr{S}$ we have $\int h d\mu \leqq \int \varphi d\mu \leqq c(\varphi)$ for all $\varphi \in \Phi$, and hence $\int h d\mu \leqq \inf c(\varphi) = c(h)$ by Def. 4.2. This shows that $c'(h) \leqq c(h)$.

To prove the converse inequality $c'(h) \geqq c(h)$ we denote for every $\varphi \in \Phi$ by μ_φ a measure in $\mathscr{S}$ such that $\mu_\varphi(\varphi) = c(\varphi)$. Since $\mathscr{S}$ is vaguely compact, there exists a cluster point $\mu \in \mathscr{S}$ for the net $(\mu_\varphi)_{\varphi \in \Phi}$. For any $\psi \in \Phi$ we obtain along Φ

$$c(h) = \lim c(\varphi) = \lim \mu_\varphi(\varphi) \leqq \lim\inf \mu_\varphi(\psi) \leqq \mu(\psi)$$

because we may restrict the attention to functions $\varphi \leqq \psi$ in Φ. It follows that

$$c(h) \leqq \inf\{\mu(\psi) \mid \psi \in \Phi\} = \mu(h) \leqq c'(h).$$

Next we obtain for any $f \in \mathscr{F}^+$

$$\begin{aligned} c_*(f) &= \sup\{c(h) \mid h \in \mathscr{H}_0,\ h \leqq f\} \\ &= \sup_{h \leqq f} \sup_{\mu \in \mathscr{S}} \int h d\mu = \sup_{\mu \in \mathscr{S}} \sup_{h \leqq f} \int h d\mu \\ &= \sup_{\mu \in \mathscr{S}} \int_* f d\mu. \\ c^*(f) &= \inf\{c_*(g) \mid g \in \mathscr{G},\ g \geqq f\} \\ &= \inf_{g \geqq f} \sup_{\mu \in \mathscr{S}} \int_* g d\mu \geqq \sup_{\mu \in \mathscr{S}} \inf_{g \geqq f} \int g d\mu \\ &= \sup_{\mu \in \mathscr{S}} \int^* f d\mu. \ ▌ \end{aligned}$$

Note that the sign of equality need not hold in (20), cf. § 5.6.

Remark. If the vaguely compact set $\mathscr{S} \subset \mathscr{M}^+$ is *hereditary* (Def. 5.2), the lower capacity $c_*(f)$ of $f \in \mathscr{F}^+(X)$ is even the supremum of $\int_* f d\mu$ as μ ranges over the smaller set consisting of those measures $\mu \in \mathscr{S}$ whose support is compact and contained in

$$A := \{x \in X \mid f(x) > 0\}.$$

In fact, for any $t < c_*(f)$ there exists $h \in \mathscr{H}_0$ such that $h \leqq f$, $c(h) > t$; and hence there exists $\mu \in \mathscr{S}$ such that $\int h d\mu > t$. The sets

$$K_n := \{x \in X \mid h(x) \geqq 1/n\}$$

are compact and contained in A. The trace μ_n of μ on K_n belongs to $\mathscr{S}$ when $\mathscr{S}$ is hereditary. Clearly

$$\int_* f d\mu_n \geqq \int h d\mu_n = \int_{K_n} h d\mu \to \int h d\mu > t$$

as $n \to \infty$. Hence $\int_* f d\mu_n > t$ for n sufficiently large.

5.6. *Example.* Let $X = \boldsymbol{R}^2 =$ the xy-plane; $\mathscr{S} = \{m_x \mid x \in \boldsymbol{R}\}$, where m_x denotes linear Lebesgue measure on the line $\{x\} \times \boldsymbol{R}$. The enveloping capacity c is given by

$$c(h) = \max_{x \in \boldsymbol{R}} \int h(x,y) dy, \qquad h \in \mathscr{C}_0^+(\boldsymbol{R}^2),$$

and similarly for the extension of c to $\mathscr{H}_0$ in view of (18). Now put $f = 1_A$, where A denotes the union of the compact sets

$$A_0 = \{0\} \times [1,2], \qquad A_n = \{1/n\} \times [0,1]$$

for $n = 1, 2, \ldots$. Then we obtain from (19) in view of Theorem 4.7 (with the weight function 1)

$$c_*(A) = c_*(f) = \sup_{x \in \boldsymbol{R}} \int_* f(x,y) dy = \sup_{n \geqq 0} m(A_n) = 1.$$

Any open set $G \supset A$ contains A_0 and hence also the segment $\{1/n\} \times [1,2]$ for some $n \in \boldsymbol{N}$. Thus $G \supset \{1/n\} \times [0,2]$ for some n, and so $c_*(G) \geqq m([0,2]) = 2$. It follows that

$$c^*(A) = c^*(f) \geqq 2 \geqq \sup_{x \in \boldsymbol{R}} \int^* f(x,y) dy \;\; (= 1).$$

The set A of class $\mathscr{K}_\sigma$ is, therefore, *not capacitable* with respect to c (although A is universally measurable). It follows that c is *not* sequentially order continuous from below. This also appears directly since A is the union of the increasing sequence of compact sets $K_n = A_0 \cup A_1 \cup \ldots \cup A_n$ with $c(K_n) = 1$, whereas $c^*(A) \geqq 2$ (actually, $c^*(A) = 2$).

5.7. *Example* (Newtonian capacity). Let $X = \boldsymbol{R}^3$, and consider the newtonian kernel

$$G(x,y) = \frac{1}{|x-y|}, \qquad x, y \in X.$$

The potential $G\mu$ of a measure $\mu \in \mathscr{M}^+$ is defined by

$$G\mu(x) = \int G(x,y)\, d\mu(y), \qquad x \in X.$$

The energy of $\mu \in \mathscr{M}^+$ is defined as $\int G\mu d\mu$. Let

$$\mathscr{S} = \{\mu \in \mathscr{M}^+ \,|\, \int G\mu d\mu \leqq 1\},$$
$$\mathscr{S}_1 = \{\mu \in \mathscr{M}^+ \,|\, G\mu \leqq 1 \text{ everywhere}\}.$$

Then $\mathscr{S}$ and $\mathscr{S}_1$ are hereditary, convex,[13] and vaguely compact. The enveloping capacities

$$c = \sup_{\mu \in \mathscr{S}} \mu, \qquad c_1 = \sup_{\mu \in \mathscr{S}_1} \mu$$

are called the newtonian *energy capacity*, resp. the ordinary newtonian capacity. The associated upper capacities c^* and c_1^* are both sequentially order continuous from below (Choquet [10] in the typical case of sets). It is a well known consequence of the maximum principle for newtonian potentials that

$$c(K)^2 = c_1(K) \qquad \text{for every compact set } K.$$

This quantity is the classical capacity of K. There is of course a similar relation between the inner, resp. outer, capacities (of arbitrary sets) associated with c and c_1. For the potential $G\mu \in \mathscr{G}$ of a measure $\mu \in \mathscr{M}^+$ we have

$$c(G\mu)^2 = \int G\mu d\mu, \qquad c_1(G\mu) = \int d\mu.$$

We refer to a forthcoming general discussion of these two types of capacity (energy capacity and usual capacity) for very general kernels G. (See also [14], [15], [16], and § 6.7 below.)

6. More about the Classes $\mathscr{G}^*$ and $\mathscr{H}_0^*$

In this section $\mathscr{S}$ denotes a *hereditary* (see end of § 5.2) and vaguely *compact* set of positive measures, and $c = \sup_{\mu \in \mathscr{S}} \mu$ the *enveloping capacity*

[13]) The convexity of $\mathscr{S}$ follows from the positive definite character of the kernel G.

as defined in (15), § 5.4. According to the representation theorem (§ 5.3) the results obtained are of course applicable to any capacity c (in the sense of Def. 4.1), taking for $\mathscr{S}$ e.g. the set $\mathscr{S}_c$ of *all* measures $\mu \leqq c$. We shall continue the study of the classes $\mathscr{G}^*$ and $\mathscr{H}_0^*$ as defined in § 2.1, now with $C = c^*$, the upper capacity associated with c. (See also Lemma 3.2.)

6.1. **Lemma.** *For any measure $\mu \in \mathscr{S}$ the μ-integrable (resp. μ-measurable, or μ-capacitable) functions $X \to [0, +\infty]$ form a closed subset of $\mathscr{F}^+(X)$ in the c^*-metric topology.*

Proof. It is well known that each of these 3 subsets of $\mathscr{F}^+(X)$ is closed in the μ^*-metric topology determined by the (pseudo)distance $\int^* |f_1 - f_2| d\mu$ between functions $f_1, f_2 \in \mathscr{F}^+(X)$. (In the case of the μ-capacitable functions, that is, functions $f \in \mathscr{F}^+(X)$ such that $\int_* f d\mu = \int^* f d\mu$, this fact is also a special case of Lemma 4.6.) Hence the present lemma follows from (20), § 5.5, according to which the μ^*-distance is majorized by the c^*-distance when $\mu \in \mathscr{S}$:

$$\int^* |f_1 - f_2| d\mu \leqq c^*(|f_1 - f_2|). \blacksquare$$

Corollary. *Any function of class $\mathscr{G}^*$ is μ-measurable and μ-capacitable for every $\mu \in \mathscr{S}$. Any function of class $\mathscr{H}_0^*$ is μ-integrable for every $\mu \in \mathscr{S}$.*

In fact, the functions of class $\mathscr{G}$ are l.s.c., hence universally capacitable and universally measurable, and the functions of class $\mathscr{H}_0$ are universally integrable, that is, integrable with respect to every (Radon) measure on X.

6.2. **Theorem.** *If $f \in \mathscr{G}^*$ (resp. $f \in \mathscr{H}_0^*$) then the mapping $\mu \mapsto \int f d\mu$ of $\mathscr{S}$ into $[0, +\infty]$ is l.s.c. (resp. u.s.c. and finite valued). The converse implication is valid under the additional hypothesis that $f \in \mathscr{H}_0^*$ (resp. $f \in \mathscr{G}^*$).*

Proof. For the applications of the converse implication in potential theory (cf. § 6.7 below) the second case $f \in \mathscr{H}_0^*$ is of particular importance, and so we shall give the proof for this case, the case $f \in \mathscr{G}^*$ being quite analogous.

First suppose that $f \in \mathscr{H}_0^*$, and choose for any $\varepsilon > 0$ a function $h \in \mathscr{H}_0$ so that $h \leqq f$ and $c^*(f - h) < \varepsilon$ (Lemma 3.2). For every $\mu \in \mathscr{S}$ we obtain from (20), § 5.5, since f and h are μ-integrable (Cor. to Lemma 6.1):

$$\int h d\mu \leqq \int f d\mu = \int h d\mu + \int (f - h) d\mu$$
$$\leqq \int h d\mu + c^*(f - h) < \int h d\mu + \varepsilon.$$

The mapping $\mu \mapsto \int f d\mu$ of $\mathscr{S}$ into $[0, +\infty[$ has thus been approximated uniformly on $\mathscr{S}$ by mappings $\mu \mapsto \int h d\mu$ with $h \in \mathscr{H}_0$, and these latter mappings are finite and u.s.c. (on all of $\mathscr{M}^+$) as observed in the beginning of the proof of Theorem 5.5.

Conversely, suppose that $f \in \mathscr{G}^*$ and that the (upper) integral $\int f d\mu$ is a finite valued u.s.c. function of $\mu \in \mathscr{S}$. (Note that $f \in \mathscr{G}^*$ is indeed μ-measurable and μ-capacitable for every $\mu \in \mathscr{S}$, again according to Cor. to Lemma 6.1.) Given $\varepsilon > 0$, there exists according to Lemma 3.2 a function $g \in \mathscr{G}$ such that $g \geqq f$ and $c^*(g - f) < \varepsilon$. Denote by Φ the upward directed family of all functions $\varphi \in \mathscr{C}_0^+$ such that $\varphi \leqq g$. Considered as functions of $\mu \in \mathscr{S}$, the integrals $\int \varphi d\mu$, $\varphi \in \Phi$, form an upward directed family of finite valued continuous functions on the compact space $\mathscr{S}$. It follows e.g. from Theorem 4.3 (a) applied to μ that

$$\sup_{\varphi \in \Phi} \int \varphi d\mu = \int g d\mu \geqq \int f d\mu$$

for every $\mu \in \mathscr{S}$. By hypothesis the function $\mu \mapsto \int f d\mu$ on $\mathscr{S}$ is finite valued and u.s.c. It follows from Dini's theorem that there exists $\varphi \in \Phi$ such that $\int \varphi d\mu > \int f d\mu - \varepsilon$ for all $\mu \in \mathscr{S}$, and hence

$$c_*((f-\varphi)^+) = \sup_{\mu \in \mathscr{S}} \int (f-\varphi)^+ d\mu \leqq \varepsilon \text{ [14]}$$

in view of (19), § 5.5. Now $f \in \mathscr{G}^*$, and $\varphi \in \mathscr{C}_0^+ \subset \mathscr{H}_0 \subset \mathscr{H}_0^*$. Thus it follows from Lemma 2.3 that $(f-\varphi)^+$ is of class $\mathscr{G}^*$ and hence capacitable (Lemma 4.6). We conclude that $c^*((f-\varphi)^+) \leqq \varepsilon$ and hence $c^*(|f-\varphi|) < 2\varepsilon$ because

$$|f-\varphi| = (f-\varphi)^+ + (\varphi - f)^+ \leqq (f-\varphi)^+ + (g-f).$$

Consequently, f belongs to the closure of $\mathscr{C}_0^+$ in the c^*-metric topology on $\mathscr{F}^+(X)$, in particular $f \in \mathscr{H}_0^*$ by definition (§ 2.1). ∎

Remark. Actually, the latter part of the proof shows that $\mathscr{G}^* \cap \mathscr{H}_0^*$ is contained in the c^*-metric closure of $\mathscr{C}_0^+$, and this is the non-trivial part of Theorem 3.3 (for the case $C = c^*$).

Corollary. *For any quasi closed set $H \subset X$ the set of all measures $\mu \in \mathscr{S}$ carried by H is vaguely compact.*

In fact, $f := 1_{\complement H} \in \mathscr{G}^*$ (Lemma 2.4), and so $\{\mu \in \mathscr{S} \mid \int f d\mu = 0\}$ is vaguely closed, hence vaguely compact, on account of the first part of the theorem (for the case $f \in \mathscr{G}^*$).

[14]) This inequality follows from the fact that, for any $\mu \in \mathscr{S}$, the trace ν of μ on the μ-measurable set $\{x \in X \mid f(x) \geq \varphi(x)\}$ belongs to $\mathscr{S}$ because $\mathscr{S}$ is hereditary. Hence we obtain

$$\int (f-\varphi)^+ d\mu = \int (f-\varphi) d\nu < \varepsilon.$$

6.3. **Theorem.** *For any function* $f \in \mathscr{H}_0^*$ *we have*

$$c(f) = \max_{\mu \in \mathscr{S}} \int f d\mu < +\infty.$$

The class $\mathscr{S}(f)$ *of all measures* $\mu \in \mathscr{S}$ *such that* $\int f d\mu = c(f)$, *is vaguely compact. It is convex if* $\mathscr{S}$ *is convex.*

Proof. The functions of class $\mathscr{H}_0^*$ are c-capacitable (Lemma 4.6) and μ-integrable for every $\mu \in \mathscr{S}$ (Cor. to Lemma 6.1). Since $\mathscr{S}$ is compact it follows from the first part of Theorem 6.2 that the supremum $c(f) = c_*(f)$ in (19) of Theorem 5.5 is indeed attained and finite, and that $\mathscr{S}(f)$ is compact. If $\mathscr{S}$ is convex then any convex combination μ of measures $\mu_1, \mu_2 \in \mathscr{S}(f)$ belongs to $\mathscr{S}$ and gives the maximal value $\int f d\mu = c(f)$. ∎

6.4. **Lemma.** *Consider a decreasing sequence of functions* $f_n \in \mathscr{H}_0^*$, *and choose corresponding maximizing measures* $\mu_n \in \mathscr{S}(f_n)$. *Then every vague cluster point for the sequence* (μ_n) *belongs to* $\mathscr{S}(\inf_n f_n)$.

Proof. Write $f = \inf_n f_n$, and let μ denote any vague cluster point for (μ_n). For any $m \in \boldsymbol{N}$ we have by Theorem 6.2

$$\int f_m d\mu \geqq \liminf_n \int f_m d\mu_n \geqq \lim_n \int f_n d\mu_n = \lim_n c(f_n) = c(f).$$

It follows that $\mu \in \mathscr{S}(f)$ because $\mu \in \mathscr{S}$ and

$$\int f d\mu = \inf_m \int f_m d\mu \geqq c(f). \ ∎$$

Remark. There is a similar result for any downward directed family $(f_\alpha)_{\alpha \in I}$ of u.s.c. functions f_α of class $\mathscr{H}_0^*$ (cf. Theorem 3.6 (b)). The proof is quite similar. In both cases the method of proof actually leads to a slightly stronger formulation. Thus, in the latter case, any vague cluster point μ for the filter on $\mathscr{S}$ generated by the "sections" $\bigcup_{\alpha \geqslant \beta} \mathscr{S}(f_\alpha)$, $\beta \in I$, belongs to $\mathscr{S}(\inf_\alpha f_\alpha)$.

6.5. We shall call the given hereditary (and vaguely compact) subset $\mathscr{S}$ of $\mathscr{M}^+$ *strictly hereditary* if, for every $\mu \in \mathscr{S}$ and every $\nu \in \mathscr{M}^+$ with $\nu \leqq \mu$, $\nu \neq \mu$, there exists a number $t > 1$ such that $t\nu \in \mathscr{S}$.

Lemma. *Let* $f \in \mathscr{H}_0^*$. *There always exist measures* $\mu \in \mathscr{S}(f)$ *concentrated on* $\{x \in X \mid f(x) > 0\}$. *If* $\mathscr{S}$ *is strictly hereditary and if* $c(f) > 0$, *then every measure* $\mu \in \mathscr{S}(f)$ *is concentrated on this set.*

Proof. Let $\mu \in \mathscr{S}(f)$, and let ν denote the trace of μ on $A := \{x \in X \mid f(x) > 0\}$. Then $\nu \in \mathscr{S}(f)$ because $\mathscr{S}$ is hereditary and $\int f d\nu = \int_{*} f \cdot 1_A d\mu = \int f d\mu$. Suppose now that $\mathscr{S}$ is strictly hereditary and that $c(f) > 0$. For every $t > 1$ we then have

$$\int f d(t\nu) = t \int f d\nu = tc(f) > c(f),$$

and hence $t\nu \notin \mathscr{S}$. Consequently $\mu = \nu$, that is, μ is concentrated on A. ▌

6.6. *The case of sets.* We shall now specialize some of the results of the present section to the case of (indicator functions for) subsets of X. In that case the compactness of $\mathscr{S}$ can be weakened to closedness at the expense of a single precaution to be observed. This will appear from the following discussion which is largely independent of the preceding theory, but contained therein as a special case whenever $\mathscr{S}$ is compact.

Thus let $\mathscr{S}$ denote any *hereditary* and vaguely *closed* subset of $\mathscr{M}^+$. For any compact set $K \subset X$ define the capacity $c(K)$ by

$$c(K) = \sup_{\mu \in \mathscr{S}} \mu(K) = \sup\{\mu(X) \mid \mu \in \mathscr{S},\ \operatorname{supp}\mu \subset K\}.$$

The identity between these two suprema follows from the assumption that $\mathscr{S}$ be hereditary (cf. remark to Theorem 5.5). Clearly each of the two suprema is attained provided that $c(K) < +\infty$ (this is the precaution alluded to above.) Note that $\mathscr{S}$ is compact if and only if $c(K) < +\infty$ for all compact sets $K \subset X$. In that case the above definition of $c(K)$ agrees with (18), § 5.5, applied to $h = 1_K \in \mathscr{H}_0$.

We denote by $\mathscr{K} = \mathscr{K}(X)$ the class of all compact subsets of X. The mapping $c: \mathscr{K} \to [0, +\infty]$ defined above is increasing and order continuous from above. The latter assertion means that

$$c(\bigcap_{\alpha} K_\alpha) = \inf c(K_\alpha)$$

for every downward directed family of compact sets K_α.[15] Since X is locally compact it follows that c is continuous from the right and hence is a capacity in the original sense of Choquet [10, § 15]. Moreover, this capacity $c: \mathscr{K} \to [0, +\infty]$ is subadditive, and $c(\emptyset) = 0$.

[15]) To prove this, let $t < \inf c(K_\alpha)$. For each α let $\mu_\alpha \in \mathscr{S}$, $\operatorname{supp}\mu_\alpha \subset K_\alpha$, and $\mu_\alpha(X) \geqq t$. Replacing, if necessary, μ_α by $(t/\mu_\alpha(X))\mu_\alpha$ (which belongs to $\mathscr{S}$ since $\mathscr{S}$ is hereditary) we may assume that $\mu_\alpha(X) = t$. Denoting by μ any vague cluster point for (μ_α), we obtain $\operatorname{supp}\mu \subset K_\beta$. for each β, and hence $\operatorname{supp}\mu \subset \bigcap K_\alpha$. Since $\mathscr{S}$ is closed, it follows that $\mu \in \mathscr{S}$, and we conclude (taking a fixed index β) that

$$c(\bigcap K_\alpha) \geqq \mu(X) = \mu(K_\beta) \geqq \liminf_\alpha \mu_\alpha(K_\beta) = t$$

because the mapping $\nu \mapsto \nu(K_\beta)$ is u.s.c., and $\mu_\alpha(K_\beta) = \mu_\alpha(X)$ for $\alpha \succcurlyeq \beta$.

With the capacity c we associate in the usual way the inner capacity c_* and the outer capacity c^* defined for arbitrary sets $A \subset X$ by

$$c_*(A) = \sup\{c(K) | K \text{ compact}, K \subset A\},$$
$$c^*(A) = \inf\{c_*(G) | G \text{ open}, G \supset A\},$$

and we may write $c(A)$ in place of $c_*(A)$ or $c^*(A)$ whenever A is capacitable, that is, $c_*(A) = c^*(A)$. Compact sets and open sets are capacitable. As in the proof of (19), § 5.5, we obtain the following representations of the inner capacity of a set $A \subset X$:

$$\left.\begin{aligned} c_*(A) &= \sup\{\mu_*(A) \mid \mu \in \mathscr{S}\} \\ &= \sup\{\mu(X) \mid \mu \in \mathscr{S}, \operatorname{supp}\mu \text{ compact and } \subset A\}. \end{aligned}\right\} \quad (21)$$

Again the identity between these suprema follows from the assumption that $\mathscr{S}$ be hereditary. Using the former representation (21) we see that the inner capacity c_* is countably subadditive on $\mathscr{S}$-measurable sets, and sequentially order continuous from below on such sets (cf. the analogous proof of Theorem 7.1 below). It follows that the outer capacity c^* is countably subadditive. In particular the quasi topological notions and results of [17] are available.

Any *quasi compact* set A is capacitable (cf. the proof of Lemma 4.6). If $c(A) < +\infty$ the former supremum in (21) is attained by some measure $\mu \in \mathscr{S}$ concentrated on the quasi compact set A. Thus

$$c(A) = \max\{\mu(X) \mid \mu \in \mathscr{S}, \mu \text{ conc. on } A\}.$$

The set $\mathscr{S}(A)$ of all maximizing measures is vaguely compact (and convex if $\mathscr{S}$ is convex).

To prove this, note that the mapping $\mu \mapsto \mu(A)$ of $\mathscr{S}$ into $[0, +\infty]$ is finite valued and u.s.c. when A is quasi compact. (This is easily reduced to the case of a compact set, cf. the proof of the first part of Theorem 6.2.) Similarly, the mapping $\mu \mapsto \mu(B)$ of $\mathscr{S}$ into $[0, +\infty]$ is l.s.c. for any quasi open set B. When applied to $B = \complement A$ this latter observation implies that those measures $\mu \in \mathscr{S}$ which are concentrated on A form a vaguely closed subset of $\mathscr{M}^+$ when A is quasi compact (or just quasi closed), cf. Cor. to Theorem 6.2. Finally, this vaguely closed set of measures is vaguely bounded, and hence vaguely compact, because

$$\mu(X) = \mu(A) \leqq c(A) < \infty$$

for every measure in the set.

6.7. *An application.* We return to the example in § 5.7, the newtonian energy capacity c. Denote by $\mathscr{E}^+$ the class of all measures $\lambda \in \mathscr{M}^+$ such that $\int G\lambda d\lambda < +\infty$. It was proved by CARTAN [9, p. 238] that, for each $\lambda \in \mathscr{E}^+$, the mapping $\mu \mapsto \int G\lambda d\mu$ of $\mathscr{S} = \{\mu \in \mathscr{M}^+ | \int G\mu d\mu \leqq 1\}$ into $[0, +\infty]$ is vaguely continuous (and finite valued). Since $G\lambda \in \mathscr{G} \subset \mathscr{G}^*$, this result, by Theorem 6.2, is equivalent to stating that $G\lambda \in \mathscr{H}_0^*$ for every $\lambda \in \mathscr{E}^+$.

Consider now a function $f \in \mathscr{H}_0^*$. It follows from the energy principle (the strict positive definite character of the newtonian kernel G) that $\mathscr{S}(f)$ consists of precisely one measure μ. By the Gauss variational method it is shown that the measure $\lambda := c(f)\mu$ is characterized within $\mathscr{E}^+$ by the following two properties

(a) $G\lambda \geqq f$ q.e.
(b) $G\lambda = f$ almost everywhere with respect to λ.

Moreover, λ is concentrated on $\{x \in X | f(x) > 0\}$ (Lemma 6.5), and $\int f d\lambda = \int G\lambda d\lambda = c(f)^2$. This measure $\lambda = \lambda_f$ is called the *capacitary measure* for the function $f \in \mathscr{H}_0^*$.

Next it is shown that, for any function $f \in \mathscr{F}^+(X)$, we have the following *dual representation* of the upper energy capacity:

$$c^*(f) = \inf\{(\int G\lambda d\lambda)^{\frac{1}{2}} | \lambda \in \mathscr{E}^+, G\lambda \geqq f \text{ q.e.}\}.$$

This allows us to deduce that the quasi u.s.c. envelope f^* of f (which exists and is uniquely determined q.e. according to [17, Theorem 3.5]) is of class $\mathscr{H}_0^*$ if and only if $c^*(f) < +\infty$. In the affirmative case we have $c(f^*) = c^*(f)$, and the above infimum is attained by precisely one measure, viz. the capacitary measure $\lambda = \lambda_{f^*}$ for f^*. We call this measure the *upper capacitary measure* for f. (Its potential is also characterized as the smallest, $\hat{R}_f$, among all potentials majorizing f quasi everywhere.)

Specializing to the case $f = 1_A$, the indicator function of a set $A \subset X$ with $c^*(A) < +\infty$, we thus obtain the *outer equilibrium* measure $\lambda = \lambda_{A^*}$, characterized within $\mathscr{E}^+$ by the properties that λ is concentrated on the (quasi compact) quasi closure A^* of A, and that

(a) $G\lambda = 1$ q.e. in A (even in A^*),
(b) $G\lambda \leqq 1$ everywhere (by the maximum principle).

Moreover, $\lambda(X) = \int G\lambda d\lambda = c^*(A)^2 (= c(A^*)^2)$.

A further important case is that of the *outer balayage* of a given measure $\mu \in \mathscr{M}^+$ on a set $A \subset X$. Here we take $f = G\mu \cdot 1_A$ and assume again that $c^*(f) < +\infty$ (e.g. $\mu \in \mathscr{E}^+$). Since $G\mu$ is always quasi continuous, we have again $f^* = G\mu \cdot 1_{A^*} \in \mathscr{H}_0^*$. We thus obtain the outer swept-out measure $\lambda = \lambda_{G\mu \cdot 1_{A^*}}$ of μ on A, characterized within $\mathscr{E}^+$ by the properties that λ is concentrated on the quasi closure A^* of A, and that

(a) $G\lambda = G\mu$ q.e. in A (even in A^*),
(b) $G\lambda \leqq G\mu$ everywhere (by the domination principle).

Moreover, $\int G\mu d\lambda = \int G\lambda d\lambda = c^*(G\mu \cdot 1_A)^2$.

In view of the compatibility between the "quasi topology" and the fine topology e.g. in the present newtonian case (cf. [17, §§ 4,5]), the results mentioned above for the two particular cases (outer equilibrium and outer balayage) coincide with those obtained by Cartan in his fundamental treatise [9] of the newtonian potential, except that our method is limited to the case $c^*(A) < +\infty$, resp. $c^*(G\mu \cdot 1_A) < +\infty$. On the other hand the present method is applicable to a very large class of kernels (consistent kernels), see [14], [15], and a comprehensive exposition to appear.

7. More about the Lower Capacity

We continue the study of a capacity c represented as the upper envelope of a hereditary and vaguely compact set $\mathscr{S}$ of positive measures on the locally compact space X. According to Theorem 5.5 the associated lower capacity c_* is given by

$$c_*(f) = \sup_{\mu \in \mathscr{S}} \int_* f d\mu \tag{19}$$

for every $f \in \mathscr{F}^+(X)$. By the remark to this theorem it suffices here to let μ range over the set of all measures $\mu \in \mathscr{S}$ of compact support contained in $\{x \in X | f(x) > 0\}$.

For brevity we shall say that a function f, or a set A, is *$\mathscr{S}$-measurable* if it is μ-measurable for every $\mu \in \mathscr{S}$.

It is possible to develop a theory for the lower capacity c_* analogous to that of Chapter I for the upper capacity $C = c^*$. In particular one may study the closed classes $\mathscr{G}_*$, $\mathscr{H}_*$, and $\mathscr{H}_{0*}$, replacing the c^*-metric by the analogous c_*-metric on $\mathscr{S}$-measurable functions. We shall, however, limit our attention to those properties of the lower capacity which are relevant for the potential theoretic applications we have in mind.

7.1. **Theorem.** *The lower capacity c_* is countably subadditive and sequentially order continuous from below on $\mathscr{S}$-measurable functions, that is,*

$$c_*\Big(\sum_{n \in \boldsymbol{N}} f_n\Big) \leqq \sum_{n \in \boldsymbol{N}} c_*(f_n),$$

$$[f_n \nearrow f] \Rightarrow [c_*(f_n) \to c_*(f)]$$

for any sequence $(f_n)_{n \in \boldsymbol{N}}$ *of* $\mathscr{S}$*-measurable functions* $f_n \in \mathscr{F}^+(X)$.

Proof. By application of (19), the proof is easily reduced to the case $c = \mu$ of a single measure $\mu \in \mathscr{M}^+$, considered in (12), (13) of § 5.2. ▌

Corollary. *For any sequence of* $\mathscr{S}$*-measurable sets* A_n,

$$c_*\Big(\bigcup_{n \in \boldsymbol{N}} A_n\Big) \leqq \sum_{n \in \boldsymbol{N}} c_*(A_n),$$

$$[A_n \nearrow A] \Rightarrow [c_*(A_n) \to c_*(A)].$$

7.2. *Exceptional sets determined by* c_*. In addition to the sets $E \subset X$ with $c^*(E) = 0$, the wider class of sets E such that $c_*(E) = 0$ plays a certain role in developing potential theory. According to the above corollary, the class of all $\mathscr{S}$*-measurable* sets E with $c_*(E) = 0$ is stable under countable union. In view of (19) we have for any set $E \subset X$

$$[c_*(E) = 0] \Leftrightarrow [\mu_*(E) = 0 \text{ for all } \mu \in \mathscr{S}],$$

(and similarly for a function $f \in \mathscr{F}^+(X)$ instead of the set E). If E is $\mathscr{S}$-measurable, then $c_*(E) = 0$ holds if and only if E is locally μ-negligible for every $\mu \in \mathscr{S}$.

Definition. *A property* $P[x]$ *is said to hold nearly everywhere* (French: à peu près partout) *in a set* $A \subset X$ *(abbreviated: n.e. in* A*) if* $c_*(\{x \in A \mid \text{non } P[x]\}) = 0$.

The following lemma is an immediate consequence of the preceding observations.

Lemma. *If a property* $P[x]$ *holds locally almost everywhere with respect to every measure* $\mu \in \mathscr{S}$*, then it holds nearly everywhere. The converse implication is valid provided that the exceptional set* $E := \{x \in X \mid \text{non} P[x]\}$ *is* $\mathscr{S}$*-measurable.*

7.3. **Lemma.** *Let* $f \in \mathscr{F}^+(X)$*. In order that* $f(x) = 0$ *n.e. it is necessary and sufficient that* $c_*(f) = 0$ *and that moreover* f *be* ν*-measurable for every* $\nu \in \mathscr{S}$ *of compact support contained in* $\{x \in X \mid f(x) > 0\}$.

Proof. Writing $E := \{x \in X \mid f(x) > 0\}$, we have (as in the proof of Lemma 1.3 (a))

$$f \leqq 1_E + 1_E + \ldots; \qquad 1_E \leqq f + f + \ldots. \tag{22}$$

If f, and hence E, is $\mathscr{S}$-measurable, these inequalities serve to establish that $f(x) = 0$ n.e. is equivalent to $c_*(f) = 0$ in view of Theorem 7.1.

For any $f \in \mathscr{F}^+(X)$ such that $f(x) = 0$ n.e. let $h \in \mathscr{H}_0$, $h \leqq f$. Then $h(x) = 0$ n.e., and so $c(h) = 0$. It follows that $c_*(f) = 0$. Moreover, the stated measurability condition is trivially fulfilled because the only measure ν in question is $\nu = 0$ since $c_*(E) = 0$, cf. (19) or (21).

Conversely, suppose that $c_*(f) = 0$, and that f is ν-measurable for every $\nu \in \mathscr{S}$ of compact support contained in E. The second inequality (22) then shows that $\int_* 1_E d\nu = 0$ for any such ν because $\int_* f d\nu = 0$. Hence $c_*(E) = c_*(1_E) = 0$ according to (19) or (21). ▌

7.4. **Theorem.** *Let $f_1, f_2 \in \mathscr{F}^+(X)$, and suppose that f_2 is $\mathscr{S}$-measurable. Consider the following statements:*

(i) $f_1(x) \leqq f_2(x)$ *nearly everywhere,*

(ii) $c_*((f_1 - f_2)^+) = 0,$

(iii) $\int_* f_1 d\mu \leqq \int_* f_2 d\mu$ *for every* $\mu \in \mathscr{S}$.

Then

$$\text{(i)} \underset{(\Leftarrow)}{\Rightarrow} \text{(ii)} \Leftrightarrow \text{(iii)} \Rightarrow [c_*(f_1) \leqq c_*(f_2)],$$

the implication (ii) $\Rightarrow$ (i) *being valid under the additional hypothesis that f_1 be $\mathscr{S}$-measurable, or just ν-measurable for every $\nu \in \mathscr{S}$ of compact support contained in $\{x \in X \mid f_1(x) > 0\}$.*

Proof. The implication (i) $\Rightarrow$ (ii) holds without any assumption on $f_1, f_2 \in \mathscr{F}^+(X)$ and follows, like its conditional converse, from the preceding lemma applied to $f = (f_1 - f_2)^+$.

To derive (iii) from (ii) let $h \in \mathscr{H}_0$, $h \leqq f_1$. Then

$$c_*((h - f_2)^+) \leqq c_*((f_1 - f_2)^+) = 0,$$

and hence $\int_* (h - f_2)^+ d\mu = 0$ for every $\mu \in \mathscr{S}$ according to (19). Since $(h - f_2)^+$ is μ-measurable and majorized by the μ-integrable function h, it follows that $\int^* (h - f_2)^+ d\mu = 0$, and

$$\int h d\mu \leqq \int_* f_2 d\mu + \int^* (h - f_2) d\mu = \int_* f_2 d\mu$$

on account of (4), § 4.5, applied to the measure μ. Consequently, $\int_* f_1 d\mu \leqq \int_* f_2 d\mu$.

Conversely, suppose that (iii) holds, and let $h \in \mathscr{H}_0$, $h \leqq (f_1 - f_2)^+$. For any $\mu \in \mathscr{S}$ of compact support contained in $\{x \in X \mid f_1(x) > f_2(x)\}$ we have $h + f_2 \leqq f_1$ almost everywhere with respect to μ, and hence

$$\int h d\mu + \int_* f_2 d\mu = \int_* (h + f_2) d\mu \leqq \int_* f_1 d\mu \leqq \int_* f_2 d\mu$$

by hypothesis. If $\int_* f_2 d\mu < +\infty$, this shows that $\int h d\mu = 0$. The same holds in general. In fact, the trace μ_n of μ on the μ-measurable set $E_n := \{x \in X | f_2(x) \leqq n\}$, $n \in \boldsymbol{N}$, has the same properties as required above for μ, and in addition $\int_* f_2 d\mu_n < +\infty$. Hence $\int h d\mu_n = 0$, and consequently $\int h d\mu = 0$ because the sets E_n, $n \in \boldsymbol{N}$, cover $\{x \in X | h(x) > 0\}$ (since $f_2(x) < +\infty$ for every x with $h(x) > 0$). Having thus proved that $\int h d\mu = 0$ for every $h \in \mathscr{H}_0$ such that $h \leqq (f_1 - f_2)^+$, we conclude that $\int_* (f_1 - f_2)^+ d\mu = 0$, and finally, by varying μ, that $c_*((f_1 - f_2)^+) = 0$.

Clearly (iii) implies that $c_*(f_1) \leqq c_*(f_2)$ without any hypotheses on f_1, f_2. ▮

Corollary 1. *Let $f \in \mathscr{F}^+(X)$ be $\mathscr{S}$-measurable, and let $0 < t < +\infty$. If $f(x) \geqq t$ n.e. in some set $A \subset X$, then $c_*(A) \leqq t^{-1} c_*(f)$.*

In fact, $t \cdot 1_A \leqq f$ n.e., hence $t c_*(A) = c_*(t \cdot 1_A) \leqq c_*(f)$.

Applying this result to $A = \{x \in X | f(x) = +\infty\}$, we obtain for $t \to +\infty$:

Corollary 2. *Let $f \in \mathscr{F}^+(X)$ be $\mathscr{S}$-measurable with $c_*(f) < +\infty$. Then* $f(x) < +\infty$ n.e.

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